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PROBLEMS IN DYNAMICS AND STABILITY OF SHELLS

FOREIGN TECHNOLOGY DIVISION

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# EDITED MACHINE TRANSLATION

PROBLEMS IN DYNAMICS AND STABILITY OF SHELLS

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P. M. OGIBALOV

# VOPROSY DINAMIKI I USTOYCHIVOSTI OBOLOCHEK

Izdatel'stvo
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1963

Pages — 417

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## U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

| Block | Italic | Transliteration | Block | Italic | Transliteration |
|---|---|---|---|---|---|
| А а | *А а* | A, a | Р р | *Р р* | R, r |
| Б б | *Б б* | B, b | С с | *С с* | S, s |
| В в | *В в* | V, v | Т т | *Т т* | T, t |
| Г г | *Г г* | G, g | У у | *У у* | U, u |
| Д д | *Д д* | D, d | Ф ф | *Ф ф* | F, f |
| Е е | *Е е* | Ye, ye; E, e* | Х х | *Х х* | Kh, kh |
| Ж ж | *Ж ж* | Zh, zh | Ц ц | *Ц ц* | Ts, ts |
| З з | *З з* | Z, z | Ч ч | *Ч ч* | Ch, ch |
| И и | *И и* | I, i | Ш ш | *Ш ш* | Sh, sh |
| Й й | *Й й* | Y, y | Щ щ | *Щ щ* | Shch, shch |
| К к | *К к* | K, k | Ъ ъ | *Ъ ъ* | " |
| Л л | *Л л* | L, l | Ы ы | *Ы ы* | Y, y |
| М м | *М м* | M, m | Ь ь | *Ь ь* | ' |
| Н н | *Н н* | N, n | Э э | *Э э* | E, e |
| О о | *О о* | O, o | Ю ю | *Ю ю* | Yu, yu |
| П п | *П п* | P, p | Я я | *Я я* | Ya, ya |

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

| Russian | English |
|---|---|
| sin | sin |
| cos | cos |
| tg | tan |
| ctg | cot |
| sec | sec |
| cosec | csc |
| sh | sinh |
| ch | cosh |
| th | tanh |
| cth | coth |
| sch | sech |
| csch | csch |
| arc sin | $\sin^{-1}$ |
| arc cos | $\cos^{-1}$ |
| arc tg | $\tan^{-1}$ |
| arc ctg | $\cot^{-1}$ |
| arc sec | $\sec^{-1}$ |
| arc cosec | $\csc^{-1}$ |
| arc sh | $\sinh^{-1}$ |
| arc ch | $\cosh^{-1}$ |
| arc th | $\tanh^{-1}$ |
| arc cth | $\coth^{-1}$ |
| arc sch | $\mathrm{sech}^{-1}$ |
| arc csch | $\mathrm{csch}^{-1}$ |
| rot | curl |
| lg | log |

## PREFACE

In modern structures of the most diverse types and purposes shells are very widely used and therefore, are of interest to us.

In writing this book we used materials from lectures on the course "Shells," read by the author at the Mechanical-Mathematical Department of the Moscow State University, well-known monographs, contemporary periodical literature, mostly Russian, and also the latest results obtained by the author in this field.

On the general theory of shells we have many good books: P. F. Papkovich "Structural mechanics of the ship," Part II; V. Z. Vlasov "General theory of shells," S. P. Timoshenko "Plates and shells," A. L. Gol'denveyzer "Theory of thin elastic shells," A. I. Lur'ye "Statics of thin-walled elastic shells," V. V. Novozhilov "Theory of thin shells," A. S. Vol'mir "Flexible plates and shells," Kh. M. Mushtari and K. Z. Galimov "Nonlinear theory of elastic shells," S. A. Ambartsumyan "Theory of anisotropic shells and others. However, the problems of dynamics and stability of shells are insufficiently elucidated, and our book fills this gap to a certain degree.

In the book we examine problems of oscillations of shells: their natural and parametric oscillations, panel flutter, and certain other dynamics problems, ~~in it~~ we also examine problems of stability of

shells within and beyond the limits of elasticity of their material and certain special problems of calculation of shells, among them the effects of hardening of shells through cold-hardening, heterogeneity of the material, and penetrating irradiation.

The book may be used as training aid for post-graduates and students of universities and technical institutes, who are specializing in the theory of elasticity and plasticity; it will be useful for scientific workers and engineers, studying the problems on strength.

The author expresses his gratitude to Reader M. A. Koltunov for his attentive and thorough editing of the book, and also thanks for their valuable advice the honored worker of science and technology of RSFSR, Doctor of Technical Sciences, Professor N. I. Bezukhov and Doctor of Physical and Mathematical Sciences, Professor V. V. Moskvitin. The author expresses his gratitude to his colleagues of the Chair of Theory of Elasticity of the Moscow State University, assistant I. M. Tyuneyeva for the help,given in preparing and putting into shape the manuscript; technician-experimenter S. A. Orlova and laboratory technician M. A. Trapp for participation in shaping of the book. The author will be grateful to all who will find it possible to send their wishes and remarks concerning the book.

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# CHAPTER I

## NECESSARY DATA ON SHELLS

### § 1. Definitions, Hypotheses, Geometric Elements

A body, limited by two curvilinear surfaces, the distance between which (thickness h) is small with respect to its other dimensions, is called a shell.

The surface, dividing it in half throughout its entire thickness is called its middle surface. It is assumed that everywhere, excluding certain points or lines on it, the middle surface is continuous with continuously variable tangent and curvatures, while all its geometric characteristics change very smoothly, i.e., so that during the transition from one point to another point located at a distance of the order of thickness h of the shell, they undergo a relative change of the order of h/R (R being the radius of curvature) or less.

We shall consider only shells of constant thickness. Depending on the shape of the middle surface we distinguish such types of shell as: cylindrical, conical, i.e., having the shape of developing surfaces; spherical, in the form of ellipsoids, and others, which have the shape of nondeveloping surfaces. In actual structures the most widely used shells are those having the shape of developing surfaces and most frequently, cylindrical shells.

Let us assume that we have separated an element of the middle surface of a shell, having an arbitrary outline; in a certain point m of it we draw a normal n to the surface, and if we draw through the normal n a number of planes, then at the intersection with the surface they will produce variously oriented plane curves — which are normal sections.

Upon rotation of a certain plane S, containing normal n, around its normal sections s are formed. Determining their curvature, we find that for the two curves r and t, lying in mutually perpendicular planes $\tilde{R}$ and $\tilde{T}$, the curvatures have extreme values with respect to other normal sections. Directions of curves r and t are termed main directions in the given point of surface, and corresponding curvatures are termed main curvatures.

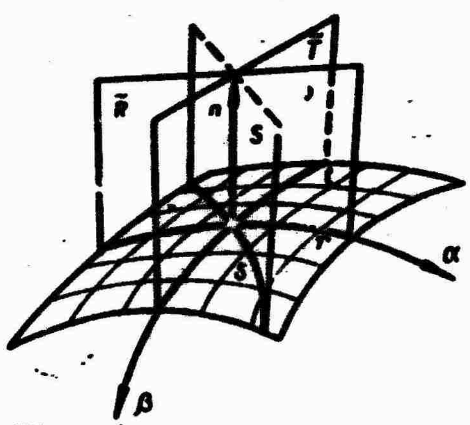

Fig. 1.

Let us assume that we found the main directions for all points of the surface, and if now we draw lines, the tangents to which in every point coincide with these directions, then we obtain lines.

For coordinate lines $\alpha$ and $\beta$ we shall select lines of main curvatures of the undeformed surface; they form an orthogonal grid on the surface (Fig. 1).

For any point of the surface we may find the Gaussian curvature, which is equal to the product of main curvatures:

$$\Gamma = k_\alpha k_\beta = k_x k_y = k_1 k_2. \tag{1.1}$$

The value

$$K = \frac{1}{2}(k_\alpha + k_\beta) = \frac{1}{2}(k_x + k_y) = \frac{1}{2}(k_1 + k_2). \tag{1.2}$$

is termed the mean curvature of the surface at the point.

Shells can be classified according to the sign of Gaussian curvature of their middle surface: thus, for example, the spherical shell has a positive Gaussian curvature, constant for all points; cylindrical and conical shells have a zero Gaussian curvature, since one of the main curvatures turns into zero, etc.

A cylindrical shell, the cross section of which is a circumference, is termed circular; where, if its section constitutes a full circumference, it will be a closed circular shell, and, if its section constitutes only a part of circumference, it will be an open circular shell.

A shell of any shape, the rise of which H is comparable to its thickness and is small when compared to its other dimensions, is usually considered to be a sloping shell.

The solution of the problem of shell equilibrium during elastic and elastoplastic deformations is based on two Kirchhoff-Love hypotheses. The first hypothesis states that the total material particles, located on the normal to the middle surface of the shell before deformation, is located also on the normal to its middle surface after deformation and, therefore, the deformed state of the shell is determined only by the deformed state of its middle surface. The second hypothesis states that all stress components which have the direction of the normal to the middle surface, are minute as compared to other stress components. These two hypotheses are in ageeement with each other and state that any thin elementary layer of material, parallel to the middle surface of the shell, is under the conditions of plane stressed state or, to be more exact, the stresses, effective in its plane, are significantly larger than other stresses.

In addition to Kirchhoff-Love hypotheses, in our research on elastoplastic deformations we shall subsequently use the assumption

of the incompressibility of the shell material. The degree of accuracy of this assumption is sufficiently definite, inasmuch as we know from the theory of elastic shells, the effect of the Poisson factor on strains and stresses. Meanwhile the incompressibility hypothesis introduces significant simplifications in the theory of elastoplastic shell deformations.

If we apply to the shell a certain relatively small distributed lateral load, then first of all, as a result of the action of compressing or stretching forces, chain stresses, evenly distributed throughout the shell's thickness will originate in the shell. Since in this case, bending stresses in the shell will be comparatively small, the shell can be termed a zero-moment shell. This is the essential feature of the shell as compared to the plane plate, which reacts to a lateral load with small sags mostly at the expense of natural bending stresses.

If the shell is sufficiently thin, then upon a further increase of the load sags, comparable with the shell's thickness, may appear in it. Then in addition to stresses in the middle surface we shall have bending stresses with them in value, and the stressed state will be of a composite or momentum nature.

Consequently, two different stressed states, which occur with small loads in cases of the plane plate and zero-moment shell, for flexible plates and shells become one composite stressed state. Hence, differential equations of the flexible plates and shells theory must have a common structure. For sloping shells, the composite stressed state is characteristic even with small loads.

When investigating, for example, the resistance of a cylindrical shell, compressed along its generatrix, we may assume that in the

initial equilibrium position the shell works as a zero-moment shell. However, upon the loss of resistance significant bending stresses immediately appear.

## § 2. Deformed State, Conditions of Compatibility

Let us select the main orthogonal system of curvilinear coordinates $\xi$, $\eta$ on the middle surface of the shell. At the point with coordinates ($\xi$ and $\eta$) of the middle surface we draw the tangent plane. We dispose of the mobile Darboux trihedron (x, y, z) in such a manner that the origin of coordinates ( $x = y = z = 0$) coincide with point ($\xi$, $\eta$), the x and y axes are directed toward the growth of $\xi$ and $\eta$, respectively and the z axis toward the center curvature of line $\xi$. Thus, the x, y axes coincide with main directions of surface at point ($\xi$, $\eta$). The middle surface element of the shell is formed by lines $\xi$ = const, $\eta$ = const and $\xi + d\xi$ = const, $\eta + d\eta$ = const, and the element of the shell — by drawing normal sections through the above-mentioned lines (Fig. 2). We designate the displacement of points of the middle surface along lines x, y, and z through u, v, and w respectively. For initial curvatures of lines x and y let us introduce the designations $k_x = k_1$, $k_y = k_2$.

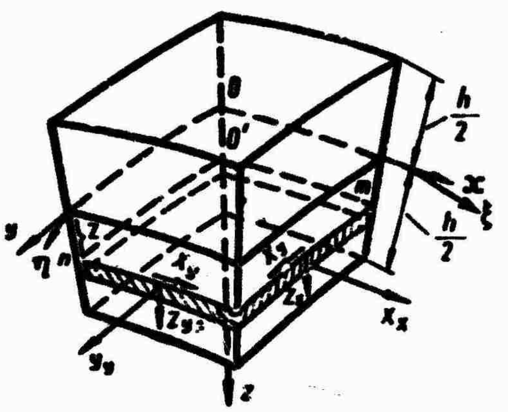

Fig. 2.

Elongations per unit length and the shift of the element of middle surface resulting from the deformation of shell we designate:

$$\varepsilon_1 = (e_{xx})_{z=0}, \quad \varepsilon_2 = (e_{yy})_{z=0}, \quad \varepsilon_{12} = \frac{1}{2}(e_{xy})_{z=0}, \tag{2.1}$$

and the change of its normal curvatures and torques, which from now on we shall term shell distortions due to strains, we respectively designate:

$$\varkappa_1 = \frac{1}{R_1'} - \frac{1}{R_1}, \quad \varkappa_2 = \frac{1}{R_2'} - \frac{1}{R_2}, \quad \tau = \frac{1}{\rho'} - \frac{1}{\rho} = \varkappa_{12}. \tag{2.2}$$

If components of the displacement vector of the point of the middle surface x, y, z axes are given as functions of coordinates $(\xi, \eta)$, then deformations $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_{12}$ are expressed through them by known formulas, containing derivatives from displacements not exceeding the first order, and distortions $\varkappa_1$, $\varkappa_2$, $\varkappa_{12}$ – not higher than the second order.

According to the first Kirchhoff-Love hypothesis, the normal element of the shell before deformation remains also normal to the middle surface after deformation, therefore, small deformations of the layer, located at the distance z from the middle surface, will be:

$$\begin{aligned} \text{Э}_{xx} &= e_{xx} = \varepsilon_1 - z\varkappa_1, \\ \text{Э}_{yy} &= e_{yy} = \varepsilon_2 - z\varkappa_2, \\ 2\text{Э}_{xy} &= e_{xy} = 2(\varepsilon_{12} - z\varkappa_{12}), \end{aligned} \tag{2.3}$$

or, expressed through displacements:

$$\begin{aligned} \text{Э}_{xx} &= e_{xx} = \frac{\partial u}{\partial x} - w\frac{\partial^2 w}{\partial x^2}, \\ \text{Э}_{yy} &= e_{yy} = \frac{\partial v}{\partial y} - w\frac{\partial^2 w}{\partial y^2}, \\ 2\text{Э}_{xy} &= e_{xy} = 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - w\frac{\partial^2 w}{\partial x \partial y}\right). \end{aligned} \tag{2.4}$$

Formulas (2.3) fully determine the rule of signs for distortions. For instance, value $\varkappa_1$ is considered positive in the case when the fiber, parallel to the x axis and located on the side of positive values z, is shortened owing to distortion $\varkappa_1$; torsion $\tau = \varkappa_{12}$ is positive, if the angle between fibers, which are parallel to x and y and located on the side of positive z, increases.

In examining the elastoplastic deformation problems we shall require an expression for deformation intensity, which we write in the form

$$e_i = \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon} = \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon - 2zP_{\varepsilon\varkappa} + z^2 P_\varkappa}\,, \tag{2.5}$$

where $P_\varepsilon$, $P_{\varepsilon\varkappa}$, $P_\varkappa$ are quadratic forms:

$$P_\varepsilon = \varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2^2 + \varepsilon_{12}^2; \tag{2.6}$$

$$P_\varkappa = \varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2 + \varkappa_{12}^2; \tag{2.7}$$

$$P_{\varepsilon\varkappa} = \varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 + \frac{1}{2}\varepsilon_1\varkappa_2 + \frac{1}{2}\varepsilon_2\varkappa_1 + \varepsilon_{12}\varkappa_{12}. \tag{2.8}$$

Expressions for deformations of the middle surface when the shell has sags comparable with its thickness, can be presented by formulas:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} - wk_x + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ \varepsilon_y &= \frac{\partial v}{\partial y} - wk_y + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \gamma &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \end{aligned} \tag{2.9}$$

Deformations (2.9), similar to (2.4), are dependent, for them the compatibility condition should be met:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma}{\partial x\,\partial y} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - k_x\frac{\partial^2 w}{\partial y^2} - k_y\frac{\partial^2 w}{\partial x^2}. \tag{2.10}$$

Let us give short systematization of data on the deformation of shells and consider the deformation during final displacements [1].

We shall determine the position of the shell point with Gaussian coordinates α, β of the middle surface of the z coordinate, directed toward the outward normal. Lamé coefficients for the system of coordinates (α, β, and z) will be equal to

$$H_1 = A\left(1 + \frac{z}{R_1}\right),\ H_2 = B\left(1 + \frac{z}{R_2}\right),\ H_3 = 1. \tag{2.11}$$

For a thin shell we can assume that

$$1 + \frac{z}{R_1} \approx 1, \quad 1 + \frac{z}{R_2} \approx 1,$$

i.e.,

$$H_1 = A,\ \ H_2 = B,\ \ H_3 = 1. \tag{2.11'}$$

Values A, B. $R_1$, $R_2$ are connected by well-known Gauss-Kodazzi

relationships:

$$\frac{\partial}{\partial\alpha}\left(\frac{B}{R_2}\right)=\frac{1}{R_1}\frac{\partial B}{\partial\alpha},$$
$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right)=-\frac{AB}{R_1R_2},\qquad(2.12)$$
$$\frac{\partial}{\partial\beta}\left(\frac{A}{R_1}\right)=\frac{1}{R_2}\frac{\partial A}{\partial\beta}.$$

In the instance of shell of rotation we can assume that the length of the arc of meridian s, is the $\alpha$ coordinate and the azimuth is the $\beta$ coordinate, then

$$A = 1,\ B = r_0(s),\qquad(2.13)$$

where $r_0(s)$ is the distance from the axis of rotation to the point with the s coordinate. In this, if $\alpha$ is the angle formed by the tangent to meridian and axis of rotation, then

$$\frac{1}{R_1}=\frac{\partial\alpha}{\partial s},\quad \frac{1}{R_2}=\frac{\cos\alpha}{r_0},\quad \frac{\partial r_0}{\partial s}=-\sin\alpha.\qquad(2.14)$$

If with the increase of S the value $r_0$ decreases, then $\alpha > 0$; with the increase of $r_0$ the angle $\alpha < 0$. We note that from (2.14) it follows that

$$\frac{\partial}{\partial s}\left(\frac{1}{R_2}\right)=-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\sin\alpha}{r_0}.\qquad(2.15)$$

Let us now assume that during deformation displacements $\tilde{u}$ and $\tilde{v}$ change through the thickness of the shell according to the linear law, but displacement $\tilde{w}$ does not change

$$\tilde{u}(\alpha,\beta,z,t)=u(\alpha,\beta,t)-z\varphi(\alpha,\beta,t),$$
$$\tilde{v}(\alpha,\beta,z,t)=v(\alpha,\beta,t)-z\psi(\alpha,\beta,t),\qquad(2.16)$$
$$\tilde{w}(\alpha,\beta,z,t)=w(\alpha,\beta,t).$$

Deformation and turn parameters will be equal to

$$e_{11}=\frac{1}{1+\frac{z}{R_1}}\left[\frac{1}{A}\frac{\partial\tilde{u}}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}\tilde{v}+\frac{\tilde{w}}{R_1}\right],$$
$$e_{22}=\frac{1}{1+\frac{z}{R_2}}\left[\frac{1}{B}\frac{\partial\tilde{v}}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}\tilde{u}+\frac{\tilde{w}}{R_2}\right].\qquad(2.17)$$

$$
\begin{aligned}
e_{33} &= \frac{\partial \tilde{w}}{\partial z};\\
e_{12} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{v}}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{u}\right) +\\
&\quad + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{u}}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{v}\right).\\
e_{1z} &= \frac{\partial \tilde{u}}{\partial z} + \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{w}}{\partial \alpha} - \frac{\tilde{u}}{R_1}\right).\\
e_{2z} &= \frac{\partial \tilde{v}}{\partial z} + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{w}}{\partial \beta} - \frac{\tilde{v}}{R_2}\right);\\
2\omega_1 &= -\frac{\partial \tilde{v}}{\partial z} + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{w}}{\partial \beta} - \frac{\tilde{v}}{R_2}\right).\\
2\omega_2 &= \frac{\partial \tilde{u}}{\partial z} - \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{w}}{\partial \alpha} - \frac{\tilde{u}}{R_1}\right).\\
2\omega_z &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{v}}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{u}\right) -\\
&\quad - \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{u}}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{v}\right).
\end{aligned}
\tag{2.17 cont'd}
$$

For a thin shell, considering (2.16), we have,

$$
\begin{gathered}
e_{11} = e_{11}^0 - zk_{11}^0,\ e_{22} = e_{22}^0 - zk_{22}^0,\ e_{12} = e_{12}^0 - zk_{12}^0,\\
e_{1z} = e_{1z}^0,\ e_{2z} = e_{2z}^0,
\end{gathered}
\tag{2.18}
$$

where

$$
\begin{aligned}
e_{11}^0 &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1},\\
e_{22}^0 &= \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_2},\\
e_{12}^0 &= \frac{1}{A}\frac{\partial v}{\partial \alpha} + \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\left(\frac{\partial A}{\partial \beta}u + \frac{\partial B}{\partial \alpha}v\right),\\
e_{1z}^0 &= \frac{1}{A}\frac{\partial w}{\partial \alpha} - \frac{u}{R_1} - \varphi,\\
e_{2z}^0 &= \frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_2} - \psi,\\
k_{12}^0 &= \frac{1}{A}\frac{\partial \psi}{\partial \alpha} + \frac{1}{B}\frac{\partial \varphi}{\partial \beta} - \frac{1}{AB}\left(\frac{\partial A}{\partial \beta}\varphi + \frac{\partial B}{\partial \alpha}\psi\right),\\
k_{11}^0 &= \frac{1}{A}\frac{\partial \varphi}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\psi,\\
k_{22}^0 &= \frac{1}{B}\frac{\partial \psi}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\varphi.
\end{aligned}
\tag{2.19}
$$

In axisymmetric deformations of the shell of rotation its state will depend only on one coordinate $\alpha$, and the displacement v will be equal to zero, therefore, $e_{12} = e_{2z} = \omega_1 = \omega_2 = 0$.

For the axisymmetric deformation of thin shell we obtain

$$e_{11}^0 = \frac{\partial u}{\partial s} + \frac{w}{R_1}, \quad e_{22}^0 = \frac{1}{r_0}(-u\sin\alpha + w\cos\alpha), \quad e_{1z}^0 = \frac{\partial w}{\partial s} - \frac{u}{R_1} - \varphi,$$

$$2\omega_2 = -\varphi - \frac{\partial w}{\partial s} + \frac{u}{R_1}, \quad k_{11}^0 = \frac{\partial \varphi}{\partial s}, \qquad k_{22}^0 = -\frac{\varphi}{r_0}\sin\alpha. \tag{2.20}$$

Deformation components in large displacements will be expressed by the following formulas:

$$\begin{aligned}
\varepsilon_{11} &= e_{11} + \frac{1}{2}\left[e_{11}^2 + \left(\frac{1}{2}e_{12} + \omega_z\right)^2 + \left(\frac{1}{2}e_{1z} - \omega_2\right)^2\right],\\
\varepsilon_{22} &= e_{22} + \frac{1}{2}\left[e_{22}^2 + \left(\frac{1}{2}e_{12} - \omega_z\right)^2 + \left(\frac{1}{2}e_{2z} + \omega_1\right)^2\right],\\
\varepsilon_{12} &= e_{12} + e_{11}\left(\frac{1}{2}e_{12} - \omega_z\right) + e_{22}\left(\frac{1}{2}e_{12} + \omega_z\right) +\\
&\quad + \left(\frac{1}{2}e_{1z} - \omega_2\right)\left(\frac{1}{2}e_{2z} + \omega_1\right),\\
\varepsilon_{1z} &= e_{1z} + e_{11}\left(\frac{1}{2}e_{1z} + \omega_2\right) + e_{zz}\left(\frac{1}{2}e_{1z} + \omega_2\right) +\\
&\quad + \left(\frac{1}{2}e_{12} + \omega_z\right)\left(\frac{1}{2}e_{2z} - \omega_1\right),\\
\varepsilon_{2z} &= e_{2z} + e_{22}\left(\frac{1}{2}e_{2z} - \omega_1\right) + e_{zz}\left(\frac{1}{2}e_{2z} - \omega_1\right) +\\
&\quad + \left(\frac{1}{2}e_{12} - \omega_z\right)\left(\frac{1}{2}e_{1z} + \omega_2\right),\\
\varepsilon_{zz} &= e_{zz} + \frac{1}{2}\left[e_{zz}^2 + \left(\frac{1}{2}e_{1z} + \omega_2\right)^2 + \left(\frac{1}{2}e_{2z} - \omega_1\right)^2\right].
\end{aligned} \tag{2.21}$$

Expressions $\varepsilon_{11}$, $\varepsilon_{22}$, $\varepsilon_{12}$ through displacements $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ can be written in the form:

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{u}}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{v} + \frac{\tilde{w}}{R_1}\right) +\\
&+ \frac{1}{2}\left(\frac{1}{1+\frac{z}{R_1}}\right)^2\left[\left(\frac{1}{A}\frac{\partial \tilde{u}}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{v} + \frac{\tilde{w}}{R_1}\right)^2 +\right.\\
&\left. + \left(\frac{1}{A}\frac{\partial \tilde{v}}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{u}\right)^2 + \left(\frac{1}{A}\frac{\partial \tilde{w}}{\partial \alpha} - \frac{\tilde{u}}{R_1}\right)^2\right].
\end{aligned} \tag{2.22}$$

$$
\begin{aligned}
\varepsilon_{22} = {} & \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{v}}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{u} + \frac{\tilde{w}}{R_2}\right) + \\
& + \frac{1}{2}\left(\frac{1}{1+\frac{z}{R_2}}\right)^2\left[\left(\frac{1}{B}\frac{\partial \tilde{v}}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{u} + \frac{\tilde{w}}{R_2}\right)^2 + \right. \\
& \left. + \left(\frac{1}{B}\frac{\partial \tilde{u}}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{v}\right)^2 + \left(\frac{1}{B}\frac{\partial \tilde{w}}{\partial \beta} - \frac{\tilde{v}}{R_2}\right)^2\right].
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{12} = {} & \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{v}}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{u}\right) + \\
& + \frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{u}}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{v}\right) + \\
& + \frac{1}{\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)}\left[\left(\frac{1}{A}\frac{\partial \tilde{u}}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{v} + \frac{\tilde{w}}{R_1}\right) \times \right. \\
& \times \left(\frac{1}{B}\frac{\partial \tilde{u}}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{v}\right) + \left(\frac{1}{A}\frac{\partial \tilde{v}}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{u}\right) \times \\
& \times \left(\frac{1}{B}\frac{\partial \tilde{v}}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{u} + \frac{\tilde{w}}{R_2}\right) + \left(\frac{1}{A}\frac{\partial \tilde{w}}{\partial \alpha} - \frac{\tilde{u}}{R_1}\right) \times \\
& \left. \times \left(\frac{1}{B}\frac{\partial \tilde{w}}{\partial \beta} - \frac{\tilde{v}}{R_2}\right)\right].
\end{aligned}
\tag{2.22 cont'd}
$$

Substituting in (2.22) expressions (2.16) for displacements, we obtain:

$$
\begin{aligned}
\left(1+\frac{z}{R_1}\right)\varepsilon_{11} = {} & \frac{1}{A}\frac{\partial u}{\partial \alpha}\left[1 + \frac{1}{2}\,\frac{1}{1+\frac{z}{R_1}} \times \right. \\
& \left. \times \left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1}\right)\right] + \frac{1}{AB}\frac{\partial A}{\partial \beta}v \times \\
& \times \left[1 + \frac{1}{2}\,\frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1}\right)\right] + \\
& + \frac{w}{R_1}\left[1 + \frac{1}{2}\,\frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1}\right)\right] + \\
& + \frac{1}{2}\,\frac{1}{1+\frac{z}{R_1}}\left[\left(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u\right)^2 + \right. \\
& \left. + \frac{1}{A^2}\left(\frac{\partial w}{\partial \alpha}\right)^2 + \frac{u^2}{R_1^2} - 2\,\frac{1}{A}\frac{\partial w}{\partial \alpha}\frac{u}{R_1}\right] - z\left\{\frac{1}{A}\frac{\partial \varphi}{\partial \alpha}\left[1 + \right.\right. \\
& \left. + \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1}\right)\right] + \\
& + \frac{1}{AB}\frac{\partial A}{\partial \beta}\psi\left[1 + \frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1}\right)\right] +
\end{aligned}
\tag{2.23}
$$

$$+\frac{1}{1+\frac{z}{R_1}}\left[\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)\left(\frac{1}{A}\frac{\partial\psi}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}\varphi\right)-\right.$$

$$\left.\left.-\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)\frac{\varphi}{R_1}\right]\right\}+z^2O_{11},$$

$$\left(1+\frac{z}{R_2}\right)\varepsilon_{22}=\frac{1}{B}\frac{\partial v}{\partial\beta}\times$$

$$\times\left[1+\frac{1}{2}\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u+\frac{w}{R_2}\right)\right]+$$

$$+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u\left[1+\frac{1}{2}\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u+\frac{w}{R_2}\right)\right]+$$

$$+\frac{w}{R_2}\left[1+\frac{1}{2}\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u+\frac{w}{R_2}\right)\right]+$$

$$+\frac{1}{2}\frac{1}{1+\frac{z}{R_2}}\left[\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)^2+\frac{1}{B^2}\left(\frac{\partial w}{\partial\beta}\right)^2+\right.$$

$$\left.+\frac{v^2}{R_2^2}-2\frac{1}{B}\frac{\partial w}{\partial\beta}\frac{v}{R_2}\right]-z\left\{\frac{1}{B}\frac{\partial\psi}{\partial\beta}\times\right. \qquad \text{(2.23 cont'd)}$$

$$\times\left[1+\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u+\frac{w}{R_2}\right)\right]+$$

$$+\frac{1}{AB}\frac{\partial B}{\partial\alpha}\varphi\left[1+\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u+\frac{w}{R_2}\right)\right]+$$

$$+\frac{1}{1+\frac{z}{R_2}}\left[\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)\left(\frac{1}{B}\frac{\partial\varphi}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}\psi\right)-\right.$$

$$\left.\left.-\left(\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2}\right)\frac{\psi}{R_2}\right]\right\}+z^2O_{22},$$

$$\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\varepsilon_{12}=\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)\times$$

$$\times\left[\left(1+\frac{z}{R_2}\right)+\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u+\frac{w}{R_2}\right)\right]+$$

$$+\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)\left[\left(1+\frac{z}{R_1}\right)+\right.$$

$$\left.+\left(\frac{1}{A}\frac{\partial u}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}v+\frac{w}{R_1}\right)\right]+$$

$$+\left[\frac{1}{AB}\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}-\frac{u}{R_1B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2A}\frac{\partial w}{\partial\alpha}+\frac{uv}{R_1R_2}\right]-$$

$$-z\left\{\left(\frac{1}{A}\frac{\partial\psi}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}\varphi\right)\left[\left(1+\frac{z}{R_2}\right)+\frac{1}{B}\frac{\partial v}{\partial\beta}+\right.\right.$$

$$
\begin{aligned}
&+\frac{1}{AB}\frac{\partial B}{\partial \alpha}u+\frac{w}{R_2}\Big]+\left(\frac{1}{B}\frac{\partial \gamma}{\partial \beta}-\frac{1}{AB}\frac{\partial B}{\partial \alpha}\psi\right)\times \\
&\times\left[\left(1+\frac{z}{R_1}\right)+\frac{1}{A}\frac{\partial u}{\partial \alpha}+\frac{1}{AB}\frac{\partial A}{\partial \beta}v+\frac{w}{R_1}\right]+ \\
&+\left[\left(\frac{1}{B}\frac{\partial \gamma}{\partial \beta}+\frac{1}{AB}\frac{\partial B}{\partial \alpha}\varphi\right)\left(\frac{1}{A}\frac{\partial v}{\partial \alpha}-\frac{1}{AB}\frac{\partial A}{\partial \beta}u\right)+\right. \\
&+\left.\left(\frac{1}{A}\frac{\partial \gamma}{\partial \alpha}+\frac{1}{AB}\frac{\partial A}{\partial \beta}\psi\right)\left(\frac{1}{B}\frac{\partial u}{\partial \beta}-\frac{1}{AB}\frac{\partial B}{\partial \alpha}v\right)\right]- \\
&-\left[\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}-\frac{u}{R_1}\right)\frac{\psi}{R_2}+\left(\frac{1}{B}\frac{\partial w}{\partial \beta}-\frac{v}{R_2}\right)\frac{\gamma}{R_1}\right]\Big\}+z^2O_{12}.
\end{aligned}
\qquad (2.23\ \text{cont'd})
$$

Formulas for shear deformations in large displacements have the form:

$$
\begin{aligned}
\varepsilon_{1z}&=\frac{\partial \tilde{u}}{\partial z}+\frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{w}}{\partial \alpha}-\frac{\tilde{u}}{R_1}\right)+ \\
&+\frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{u}}{\partial \alpha}+\frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{v}+\frac{\tilde{w}}{R_1}\right)\frac{\partial \tilde{u}}{\partial z}+ \\
&+\frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{v}}{\partial \alpha}-\frac{1}{AB}\frac{\partial A}{\partial \beta}\tilde{u}\right)\frac{d\tilde{v}}{dz}+ \\
&+\frac{1}{1+\frac{z}{R_1}}\left(\frac{1}{A}\frac{\partial \tilde{w}}{\partial \alpha}-\frac{\tilde{u}}{R_1}\right)\frac{\partial \tilde{w}}{\partial z}, \\
\varepsilon_{2z}&=\frac{\partial \tilde{v}}{\partial z}+\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{w}}{\partial \beta}-\frac{\tilde{v}}{R_2}\right)+ \\
&+\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{v}}{\partial \beta}+\frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{u}+\frac{\tilde{w}}{R_2}\right)\frac{\partial \tilde{v}}{\partial z}+ \\
&+\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{u}}{\partial \beta}-\frac{1}{AB}\frac{\partial B}{\partial \alpha}\tilde{v}\right)\frac{\partial \tilde{u}}{\partial z}+ \\
&+\frac{1}{1+\frac{z}{R_2}}\left(\frac{1}{B}\frac{\partial \tilde{w}}{\partial \beta}-\frac{\tilde{v}}{R_2}\right)\frac{\partial \tilde{w}}{\partial z}.
\end{aligned}
\qquad (2.24)
$$

After introduction of formulas (2.16) in (2.24), we obtain,

$$
\begin{gathered}
\left(1+\frac{z}{R_1}\right)\varepsilon_{1z}=\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}-\left[\varphi\left(1+\frac{1}{A}\frac{\partial u}{\partial\alpha}+\right.\right. \\
\left.+\frac{w}{R_1}+\frac{1}{AB}\frac{\partial A}{\partial\beta}v\right)+\psi\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)\Big]- \\
-z\Big\{-\Big[\varphi\left(\frac{1}{A}\frac{\partial\varphi}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}\psi\right)+ \\
+\psi\left(\frac{1}{A}\frac{\partial\psi}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}\varphi\right)\Big]\Big\}. \qquad (2.25) \\
\left(1+\frac{z}{R_2}\right)\varepsilon_{2z}=\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2}- \\
-\Big[\psi\left(1+\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{w}{R_2}+\frac{1}{AB}\frac{\partial B}{\partial\alpha}u\right)+ \\
+\varphi\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)\Big]-z\Big\{-\Big[\psi\left(\frac{1}{B}\frac{\partial\psi}{\partial\beta}+\right. \\
\left.+\frac{1}{AB}\frac{\partial B}{\partial\alpha}\varphi\right)+\varphi\left(\frac{1}{B}\frac{\partial\varphi}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}\psi\right)\Big]\Big\}.
\end{gathered}
$$

In the future we shall consider deformations of the element of shell with the dz thickness to be small as compared to 1, and, consequently, the element of the shell with the dimensions of $d\alpha d\beta dz$ will only turn, as the result of deformation, but will remain to be the rectangular parallelepiped, and will retain its dimensions: however, by virtue of (2.16) these parallelepipeds will shift with respect to one another by the shell thickness. We will also assume that the shell is so thin that $h/R \leq 0.01$; sags w will be considered small as compared to the thickness of the shell. Considering the possibility that the shell material may exceed the elastic limits and its work until reaching the tensile strength, we should assume that deformations $\varepsilon_{ik}$ are of the order of 0.1 ($\varepsilon_{ik} \leq 0.1$).

Let us consider two of the most frequently encountered instances:

1. Nonsloping shell, i.e., when the least radius of curvature is of the order of linear dimensions of the shell.

2. Sloping shells, i.e., when the least radius of curvature is

one order larger than linear dimensions $L/R \leq 0.1$.* Moreover, in the future we shall disregard the values of the order of $w/L$ or less as compared to 1. We shall also assume that components of deformation throughout the thickness of the shell change according to the linear law, i.e.,

$$
\begin{gathered}
e_{11}(\alpha,\beta,z,t)=\varepsilon^{0}_{11}(\alpha,\beta,t)-zk_{11}(\alpha,\beta,t),\quad e_{22}(\alpha,\beta,z,t)=\varepsilon^{0}_{22}(\alpha,\beta,t)-zk_{22}(\alpha,\beta,t),\\
e_{12}(\alpha,\beta,z,t)=\varepsilon^{0}_{12}(\alpha,\beta,t)-zk_{12}(\alpha,\beta,t),\quad e_{1z}(\alpha,\beta,z,t)=\varepsilon^{0}_{1z}(\alpha,\beta,t)-zk_{1z}(\alpha,\beta,t),\\
e_{2z}(\alpha,\beta,z,t)=\varepsilon^{0}_{2z}(\alpha,\beta,t)-zk_{2z}(\alpha,\beta,t).
\end{gathered}
\tag{2.26}
$$

1. Nonsloping shells. In this case $w/R \leq 0.1$, i.e., $w/R \approx h/L$ or $w/R = 10h/L$. Let us assume that angles of inclination of tangents $\partial w/\partial s_1$ and $\partial w/\partial s_2$ are such that their squares are of the order of the deformation parameters $\left(\frac{\partial w}{\partial s}\right)^2 \approx 0.1$; consequently, $\left(\frac{\partial w}{\partial s}\right) \approx \left(\frac{h}{L}\right)^{1/2}$, i.e., the variability index for the derivative from the sag along the coordinate is equal to $(h/L)^{1/2}$. The angles of rotation $\varphi$ and $\psi$ will be of the same order. The change of angle of rotation along the coordinate lines characterizes the bend of shell. We can easily note that from expressions for deformation parameters it follows that $\partial u/\partial s$ and $\partial v/\partial s$ are of the order of $(h/L)$, i.e., $u \leq h$, $v \leq h$.

Let us examine in detail the following three forms of bend: slight, average and strong, included in the described case.

a) The slight bend of the shell occurs, when turns of its lineal elements during bending are small everywhere as compared to unity. This may take place for sags, which are small as compared to the shell

---

*The sloping shell can be defined, according to M. A. Koltunov [48], as the shell, for which $L/R \approx 10h/L$, i.e., the shell, which can snap without the appearance of plastic deformations on the boundary of the region of stability.

thickness, for which $w \lesssim h\varepsilon_p^{1/2}$. In this case in the expression for deformation of the middle surface it is necessary to retain members up to $(h/L)^{3/2}$ inclusive. Then we have

$$\varepsilon_{11}^{0} = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_1} + \frac{1}{2}\frac{1}{A^2}\left(\frac{\partial w}{\partial \alpha}\right)^2 - \frac{u}{R_1 A}\frac{\partial w}{\partial \alpha},$$

$$k_{11} = \frac{1}{A}\frac{\partial \varphi}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\psi,$$

$$\varepsilon_{22}^{0} = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_2} + \frac{1}{2}\frac{1}{B^2}\left(\frac{\partial w}{\partial \beta}\right)^2 -$$
$$- \frac{v}{R_2 B}\frac{\partial w}{\partial \beta},$$

$$k_{22} = \frac{1}{B}\frac{\partial \psi}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\varphi,$$

$$\varepsilon_{12}^{0} = \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u + \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v +$$
$$+ \frac{1}{AB}\frac{\partial w}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{u}{R_1 B}\frac{\partial w}{\partial \beta} - \frac{v}{R_2 A}\frac{\partial w}{\partial \alpha},$$

$$k_{12} = \frac{1}{A}\frac{\partial \psi}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\varphi + \frac{1}{B}\frac{\partial \varphi}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\psi. \qquad (2.27)$$

Disregarding in the expressions for shear, the value (h/L) as compared to 1, we obtain

$$\varepsilon_{1z}^{0} = \frac{1}{A}\frac{\partial w}{\partial \alpha} - \frac{u}{R_1} - \varphi,$$

$$k_{1z} = 0,$$

$$\varepsilon_{2z}^{0} = \frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_2} - \psi, \qquad (2.28)$$

$$k_{2z} = 0.$$

b) The <u>average bend</u> of shell takes place, when the sag is of the order of thickness, but significantly less than characteristic linear dimensions of the shell. For the average bend we usually disregard squares of turn of the element from the bend as compared to unity. In this case $\partial\varphi/\partial s \approx \varphi/\sqrt{Lh}$, $h\,(\partial\varphi/\partial s) \approx (h/L)^{1/2}$. Here in expressions for deformations it is possible to be limited only by the terms of the order up to (h/L) inclusive, and, consequently, nonlinear terms, containing displacements of the middle surface u and v, can be disregarded, if, however, we take into consideration terms up to $(h/L)^{3/2}$

inclusive, then formulas (2.27) will remain in force, and deformations of shear will be determined by formulas (2.28).

c) The strong bend of shell is a term given to such a bend, where sags are large as compared to its thickness and are commensurable with its linear dimensions. In this case turns of lineal elements will be commensurable with unity, but $\partial\varphi/\partial s \approx \varphi/h$, $h\,(\partial\varphi/\partial s) \approx (h/L)^{1/2}$ and, consequently, with the accuracy up to terms $(h/L)$, expressions for deformations $\varepsilon_{11}$, $\varepsilon_{22}$, $\varepsilon_{12}$ will be the same, as and in the preceding case. In shear deformations it is necessary to consider values $k_{1z}$ and $k_{2z}$, equal to

$$
\begin{aligned}
k_{1z} &= -\left[\varphi\left(\frac{1}{A}\frac{\partial\gamma}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\psi\right) + \right.\\
&\quad \left. + \psi\left(\frac{1}{A}\frac{\partial\psi}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\varphi\right)\right].\\
k_{2z} &= -\left[\psi\left(\frac{1}{B}\frac{\partial\psi}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\varphi\right) + \right.\\
&\quad \left. + \varphi\left(\frac{1}{B}\frac{\partial\gamma}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\psi\right)\right].
\end{aligned}
\tag{2.29}
$$

If, however, in expressions for deformation we disregard terms $(h/L)^{1/2} \approx 0.1$ as compared to 1, then it is possible not to consider the deformation of the middle surface, and the shear deformation can be considered constant throughout the thickness and equal to

$$
\begin{aligned}
\varepsilon_{1z} &= \overset{0}{\varepsilon}_{1z} = \frac{1}{A}\frac{\partial w}{\partial\alpha} - \varphi,\\
\varepsilon_{2z} &= \overset{0}{\varepsilon}_{2z} = \frac{1}{B}\frac{\partial w}{\partial\beta} - \psi.
\end{aligned}
\tag{2.30}
$$

The strong bend, as a rule, takes place only when the external load changes sharply on a small section of the shell surface.

It is permissible to consider, as we do for small sags, that the shear deformation does not change throughout the thickness with any variation of the angle of rotation of the normal. Thus, for sags of the nonsloping shell, which are comparable with its thickness, deformation components will be calculated by the formulas (2.26)-(2.28).

Calculation of values $k_{1z}$ and $k_{2z}$ according to (2.29) will not cause any complications.

With axisymmetric deformation of the shell of rotation components of deformation $\varepsilon_{1z}$ and $\varepsilon_{2z}$ will be equal to zero, since the displacement $\tilde{v} = 0$, and other displacements are determined by formulas:

$$\begin{gathered}
\overset{\circ}{\varepsilon}_{11} = \frac{\partial u}{\partial s} + \frac{w}{R_1} + \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2 - \frac{u}{R_1}\frac{\partial w}{\partial s}, \quad k_{11} = \frac{\partial \varphi}{\partial s}, \\
\overset{\circ}{\varepsilon}_{22} = \frac{1}{r_0}(w\cos\alpha - u\sin\alpha), \quad k_{22} = -\frac{\varphi}{r_0}\sin\alpha, \\
\varepsilon_{1z} = \overset{\circ}{\varepsilon}_{1z} = \frac{\partial w}{\partial s} - \frac{u}{R_1} - \varphi.
\end{gathered} \tag{2.31}$$

We can easily see that the effect of large sags shows only on the values of meridional extension-compression deformations of the middle surface of the shell.

2. <u>Sloping shells</u>. In this case the value h/R is minute, namely $h/R \leq 0.1(h/L)$, and here, if $l/R \approx h/l$, then $h/R \approx l^2/R^2 \approx l^2/L^2$. Since deformation parameters of the middle surface are of the order of h/R, the displacements of the middle surface will be of the order of $(l/R)h$, i.e., higher than the sag. Nonlinear terms in expressions for deformation components will be effective, if angles of inclination of tangents are such that $(\partial w/\partial s)^2 \approx h/R$, i.e., $\partial w/\partial s \approx \sqrt{h/R}$, where $\partial w/\partial s \approx h/L$ when $L/R \approx h/L$. Consequently, for the bend of sloping shells it is necessary to change to nonlinear theory for smaller angles of inclination of tangents than for the bend of nonsloping shells.

In determination of deformations of sloping shells it is possible, as in plates, to disregard nonlinear terms, containing displacements of the middle surface.

## § 3. Stressed State, Equations of Equilibrium

In Fig. 2 we represented the element of shell, on the edges of which forces act in the middle surface, and in Fig. 3 we depict the element of the middle surface of the shell and the diagram of forces acting on it; bending forces are moments and transverse forces: normally a transverse load of intensity q is applied to the element.

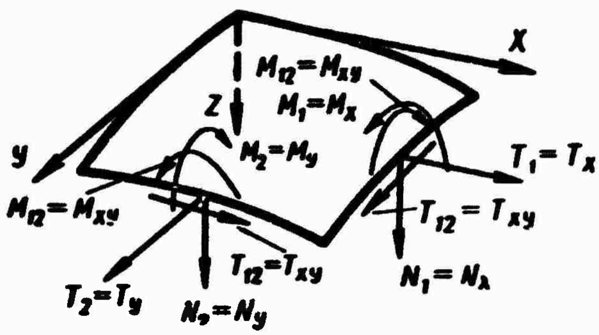


Fig. 3.

Let u. constitute the equation of equilibrium of the shell element. We write the sum of projections of all forces on the direction of tangent to line x; considering that in view of the smallness of angles, the forces in the middle surface are projected in actual size, we obtain

$$\left(X_x+\frac{\partial X_x}{\partial x}dx\right)h\,dy-X_xh\,dy+\left(X_y+\frac{\partial X_y}{\partial y}dy\right)h\,dx-$$
$$-X_yh\,dx=0.$$

Projections of transverse forces are not included here, they give terms of a higher order of smallness and therefore, can be disregarded. After simple transformations we obtain

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0. \tag{3.1}$$

Analogously we will find in projecting of all forces in the direction y, that

$$\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0. \tag{3.2}$$

The equation of moments of all forces with respect to the tangent to line y will have the form, as follows

$$\left(M_x+\frac{\partial M_x}{\partial x}dx\right)dy-M_x\,dy+\left(T_x+\frac{\partial T_x}{\partial y}dy\right)dx-$$
$$-T_x\,dx-q\,dx\,dy\frac{dx}{2}-\left(N_x+\frac{\partial N_x}{\partial x}dx\right)dy\,dx-\frac{dN_x}{\partial y}dy\,dx\frac{dx}{2}=0.$$

Leaving out small values of the highest order, we find:

$$\frac{\partial M_x}{\partial x} + \frac{\partial T_x}{\partial y} - N_x = 0. \qquad (3.3)$$

Analogously we will find the equation of moments for the tangent to line x:

$$\frac{\partial T_x}{\partial x} + \frac{\partial M_y}{\partial y} - N_y = 0. \qquad (3.4)$$

Let us now set up the equation of projections of all forces in the direction of the normal, where we consider the element of the shell in deformed state. Forces $X_x h$ and $Y_y h$ will give additional components, equal to

$$X_x h\left(k_x + \frac{\partial^2 w}{\partial x^2}\right) dx\,dy, Y_y h\left(k_y + \frac{\partial^2 w}{\partial y^2}\right) dx\,dy.$$

We can easily see that the final equation of equilibrium will have the form

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + X_x h\left(k_x + \frac{\partial^2 w}{\partial x^2}\right) + \\ + Y_y h\left(k_y + \frac{\partial^2 w}{\partial y^2}\right) + 2X_y h \frac{\partial^2 w}{\partial x\,\partial y} + q = 0. \qquad (3.5)$$

For the shell with initial deflections $w_{in}$ from the ideal form we shall have an analogous equation of equilibrium:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + X_x h\left[k_x + \frac{\partial^2}{\partial x^2}(w + w_{нч})\right] + \\ + Y_y h\left[k_y + \frac{\partial^2}{\partial y^2}(w + w_{нч})\right] + \\ + 2X_y h \frac{\partial^2}{\partial x\,\partial y}(w + w_{нч}) + q = 0.$$ [нч = in = initial] (3.6)

## § 4. Relationship Between Deformations and Stresses. Differential Equations

Let us be given a shell with thickness h, which is acted upon by a certain system of balanced forces, which cause plastic flows. Stresses in layer o'mn (Fig. 2) will be:

$$S_x = X_x - \frac{1}{2} Y_y = \frac{\sigma_i}{e_i}(\varepsilon_1 - z\varkappa_1), \qquad (4.1)$$

$$
\begin{aligned}
S_y = Y_y - \frac{1}{2} X_x = \frac{\sigma_i}{e_i} (\varepsilon_2 - z\varkappa_2), \\
S_{xy} = X_y = \frac{2\sigma_i}{3e_i} (\varepsilon_{12} - z\varkappa_{12}).
\end{aligned}
\qquad (4.1 \text{ cont'd})
$$

where $\sigma_i = \sqrt{X_x^2 - X_x Y_y + Y_y^2 + 3X_y^2}$ is the intensity of stresses; where $\sigma_i$ is a specific function $e_i$. Stresses $X_z$, $Y_z$, $Z_z$ are small when compared with basic stresses.

The entire simplification, introduced in the theory of shells by the Kirchhoff-Love hypotheses, consists of the fact that, instead of six stress components it is possible to introduce five force components and three moment components, which act on the shell element as a whole, and these eight values will be functions of only two independent variables $\xi$, $\eta$; for their determination it is sufficient to have equations of equilibrium of element only, if the relationship between forces, moments, deformations, and distortions will be established.

Five force components are determined, as resultants of all stresses along two mutually perpendicular edges of the element, lengths of arc of which in the middle surfaces are equal to unity. If the shell is sufficiently thin, so that the ratio of its thickness to the characteristic radius of curvature can be disregarded as compared with unity, then we obtain the following five expressions for the forces:

$$
\begin{aligned}
T_{xy} = T_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} X_y \, dz, \\
T_x = T_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} X_x \, dz, \quad T_y = T_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} Y_y \, dz, \\
N_x = N_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} Z_x \, dz, \quad N_y = N_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} Z_y \, dz.
\end{aligned}
\qquad (4.2)
$$

Intersecting forces $N_x$, $N_y$, in spite of the smallness of stresses, are not equal to zero, and they are determined only from equations of equilibrium.

Analogously it is possible to write formulas for the bending moments and torques:

$$M_x = M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} X_x z\,dz,\quad M_y = M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} Y_y z\,dz,$$
$$M_{xy} = M_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} X_y z\,dz. \qquad (4.3)$$

Inasmuch as stresses, applied to the element, are thus replaced by resultant forces and moments, it is possible to replace the very element of the shell (see Fig. 2) by the element of the middle surface (see Fig. 3). In Fig. 3, which shows the diagram of forces acting on the element of the middle surface of the shell, we see that forces $T_x = T_1$ and $T_y = T_2$, stretch it in the direction of x and y axes; force $T_{xy} = T_{12}$ creates a shift inside the surface, and their positive directions in x, y axes are the same, as directions of stresses $X_x$, $Y_y$, $X_y$. Positive directions of intersecting forces $N_x = N_1$, $N_y = N_2$ coincide with positive directions of stresses $Z_x$, $Z_y$. Bending moments $M_x = M_1$, $M_y = M_2$ are considered positive, if they strive to give convexity to the shell in the direction of positive z axis. Torque $M_{xy} = M_{12}$ is positive in the case when on the part of positive x axis it strives to turn the element clockwise.

For simplification of calculations, following A. A. Il'yushin [2], it is recommended instead of forces $T_1$, $T_2$, $T_{12}$, to introduce their linear combinations:

$$S_1 = T_1 - \frac{1}{2}T_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_x\,dz,$$

$$S_2 = T_2 - \frac{1}{2}T_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_y\,dz, \qquad (4.4)$$

$$\frac{2}{3} S_{12} = T_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_{xy}\,dz,$$

and instead of moments $M_1$, $M_2$, $M_{12}$ — values

$$H_1 = M_1 - \frac{1}{2}M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_x z\,dz,$$

$$H_2 = M_2 - \frac{1}{2}M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_y z\,dz, \qquad (4.5)$$

$$\frac{2}{3} H_{12} = M_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_{xy} z\,dz.$$

From (4.4) and (4.1) we now have,

$$S_1 = \varepsilon_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i}\,dz - \varkappa_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i}\,z\,dz,$$

$$S_2 = \varepsilon_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i}\,dz - \varkappa_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i}\,z\,dz, \qquad (4.4')$$

$$S_{12} = \varepsilon_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i}\,dz - \varkappa_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i}\,z\,dz,$$

and from (4.5) we obtain,

$$
\begin{aligned}
H_1 &= \varepsilon_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z\,dz - \varkappa_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2\,dz, \\
H_2 &= \varepsilon_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z\,dz - \varkappa_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2\,dz, \\
H_{12} &= \varepsilon_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z\,dz - \varkappa_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2\,dz.
\end{aligned}
\tag{4.5'}
$$

In formulas (4.4') and (4.5') we encounter three types of integrals, distributed throughtout the thickness of the shell:

$$
I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} dz, \quad I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z\,dz, \quad I_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2\,dz. \tag{4.6}
$$

Through them forces and moments are expressed as follows:

$$
\begin{aligned}
\frac{3}{4} T_1 &= \left(\varepsilon_1 + \frac{1}{2}\varepsilon_2\right) I_1 - \left(\varkappa_1 + \frac{1}{2}\varkappa_2\right) I_2, \\
\frac{3}{4} T_2 &= \left(\varepsilon_2 + \frac{1}{2}\varepsilon_1\right) I_1 - \left(\varkappa_2 + \frac{1}{2}\varkappa_1\right) I_2, \\
\frac{3}{2} T_{12} &= \varepsilon_{12} I_1 - \varkappa_{12} I_2; \\
\frac{3}{4} M_1 &= \left(\varepsilon_1 + \frac{1}{2}\varepsilon_2\right) I_2 - \left(\varkappa_1 + \frac{1}{2}\varkappa_2\right) I_3,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\frac{3}{4} M_2 &= \left(\varepsilon_2 + \frac{1}{2}\varepsilon_1\right) I_2 - \left(\varkappa_2 + \frac{1}{2}\varkappa_1\right) I_3, \\
\frac{3}{2} M_{12} &= \varepsilon_{12} I_2 - \varkappa_{12} I_3.
\end{aligned}
\tag{4.8}
$$

Since in (4.6) $\sigma_i$ is the given function of $e_i$, where its concrete form for every material becomes known in particular problems, we naturally avoid integration by z on the basis of relationship (2.5) change to integration by $e_i$. Multiplying $I_1$ by $P_\varepsilon$, $I_2$ (by $-2P_{\varepsilon\varkappa}$) and $I_3$ by $P_\varkappa$ and adding the results, we obtain:

$$I_1P_\varepsilon - 2I_2P_{\varepsilon\varkappa} + I_3P_\varkappa = \frac{3}{4}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i e_i \, dz. \qquad (4.9)$$

Differentiating (2.5) by z, we find

$$\frac{3}{4} e_i \, de_i = (zP_\varkappa - P_{\varepsilon\varkappa}) dz. \qquad (4.10)$$

Now multiplying $I_1$ by $(-P_{\varepsilon\varkappa})$, $I_2$ by $P_\varkappa$ and adding the results, we obtain:

$$-I_1P_{\varepsilon\varkappa} + I_2P_\varkappa = \frac{3}{4}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i \, de_i. \qquad (4.11)$$

Let us find expression $z^2$ through $e_i$; for this purpose it is necessary to resolve quadratic equation (2.5)

$$z^2P_\varkappa - 2zP_{\varepsilon\varkappa} + P_\varepsilon = \frac{3}{4} e_i^2,$$

the root of which, does not contradict relationship (4.10), and is

$$z = \frac{P_{\varepsilon\varkappa}}{P_\varkappa} + \frac{\sqrt{3}}{2\sqrt{P_\varkappa}}\sqrt{e_i^2 - \frac{4}{3}\frac{P_\varepsilon P_\varkappa - P_{\varepsilon\varkappa}^2}{P_\varkappa}} \operatorname{sign}(zP_\varkappa - P_{\varepsilon\varkappa}), \qquad (4.12)$$

where it is necessary to take always positive value of the square root. Differentiating (4.12), we obtain:

$$dz = \frac{\sqrt{3}}{2\sqrt{P_\varkappa}} \frac{e_i \, de_i \operatorname{sign} de_i}{\sqrt{e_i^2 - \frac{4}{3}\frac{P_\varepsilon P_\varkappa - P_{\varepsilon\varkappa}^2}{P_\varkappa}}}. \qquad (4.13)$$

Sign of value $(zP_\varkappa - P_{\varepsilon\varkappa})$,* according to (4.10), coincides with the sign $de_i/dz$, and since in the intervals which interest us dz is always positive when z changes from -h/2 to +h/2, then integration by $de_i$ should be executed in such a manner that $de_i$ increase also, i.e., we must integrate by $de_i$ sign $de_i$.

Let us examine values of intensity of deformations in three

*Designated by symbol sign.

points, located on z axis:

$$z=-\frac{h}{2},\quad z=+\frac{h}{2},\quad z=z_0. \tag{4.14}$$

where

$$z_0=\frac{P_{\epsilon x}}{P_x}.$$

We designate them respectively:

$$\begin{aligned} e_{i1}&=\frac{2}{\sqrt{3}}\sqrt{P_\epsilon+hP_{\epsilon x}+\frac{h^2}{4}P_x} &&\left(z=-\frac{h}{2}\right),\\ e_{i2}&=\frac{2}{\sqrt{3}}\sqrt{P_\epsilon-hP_{\epsilon x}+\frac{h^2}{4}P_x} &&\left(z=+\frac{h}{2}\right),\\ e_{i0}&=\frac{2}{\sqrt{3}\sqrt{P_x}}\sqrt{P_\epsilon P_x-P_{\epsilon x}^2} &&(z=z_0).\end{aligned} \tag{4.15}$$

As we see from (4.10), point $z = z_0$ is the point of minimum $e_i$, since $d^2e_i/dz^2 > 0$. Consequently, inequalities

$$e_{i1}>e_{i0},\qquad e_{i2}\geqslant e_{i0} \tag{4.15'}$$

always occur.

We assume that tensile and shearing strains of the middle surface $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_{12}$ are either commensurable or small compared with flexural strains of the shell $\pm(h/2)\varkappa_1$, $\pm(h/2)\varkappa_2$, $\pm(h/2)\varkappa_{12}$ or that the latter are dominating if point $z_0$ does not occur beyond the limits of the thickness of the shell, i.e., if

$$-\frac{h}{2}<z_0=\frac{P_{\epsilon x}}{P_x}\leqslant\frac{h}{2}. \tag{4.16}$$

Deformations of the middle surface are termed large, or dominating, as compared with flexural strains, if point $z_0$ is located outside the thickness of the shell, i.e., if one of inequalities takes place

$$z_0=\frac{P_{\epsilon x}}{P_x}>\frac{h}{2},\quad z_0=\frac{P_{\epsilon x}}{P_x}<-\frac{h}{2}. \tag{4.17}$$

In case of commensurable tensile and flexural strains the integral from any positive value R throughout the thickness of the shell must be calculated by the formula

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} R\,dz - \frac{\sqrt{3}}{2\sqrt{P_z}}\left[\int_{e_{i0}}^{e_{i1}} \frac{Re_i\,de_i}{\sqrt{e_i^2-e_{i0}^2}} + \int_{e_{i0}}^{e_{i2}} \frac{Re_i\,de_i}{\sqrt{e_i^2-e_{i0}^2}}\right]. \tag{4.16'}$$

In case of incommensurable or large tensile strains such an integral should be calculated by the formula

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} R\,dz = \frac{\sqrt{3}\,\mathrm{sign}\,(e_{i2}-e_{i1})}{2\sqrt{P_z}} \int_{e_{i1}}^{e_{i2}} \frac{Re_i\,de_i}{\sqrt{e_i^2-e_{i0}^2}}. \tag{4.17'}$$

We now introduce designations of basic values in the theory of shells:

$$A = A_0,\quad B = B_0,\quad C = C_0 \quad \left(-\frac{h}{2} < z_0 < \frac{h}{2}\right);$$
$$A = A_1,\quad B = B_1,\quad C = C_1 \quad \left(|z_0| > \frac{h}{2}\right). \tag{4.18}$$

where values $A_0$, $B_0$, $C_0$ pertain to the case of dominating flexural strains and are equal:

$$A_0 = -\int_{e_{i0}}^{e_{i1}} \sigma_i\,de_i + \int_{e_{i0}}^{e_{i2}} \sigma_i\,de_i = \int_{e_{i1}}^{e_{i2}} \sigma_i\,de_i,$$
$$B_0 = \int_{e_{i0}}^{e_{i1}} \frac{\sigma_i\,de_i}{\sqrt{e_i^2-e_{i0}^2}} + \int_{e_{i0}}^{e_{i2}} \frac{\sigma_i\,de_i}{\sqrt{e_i^2-e_{i0}^2}}, \tag{4.18'}$$
$$C_0 = \int_{e_{i0}}^{e_{i1}} \sigma_i \sqrt{e_i^2-e_{i0}^2}\,de_i + \int_{e_{i0}}^{e_{i2}} \sigma_i \sqrt{e_i^2-e_{i0}^2}\,de_i,$$

and $A_1$, $B_1$, $C_1$ pertain to the case of dominating extension of the middle surface and are determined by the formulas:

$$A_1 = A_0 = \int_{e_{i1}}^{e_{i2}} \sigma_i\,de_i,\quad B_1 = \int_{e_{i1}}^{e_{i2}} \frac{\sigma_i\,de_i}{\sqrt{e_i^2-e_{i0}^2}}\,\mathrm{sign}\,(e_{i2}-e_{i1}),$$
$$C_1 = \int_{e_{i1}}^{e_{i2}} \sigma_i \sqrt{e_i^2-e_{i0}^2}\,de_i\,\mathrm{sign}\,(e_{i2}-e_{i1}). \tag{4.18''}$$

Integrals $I_1$, $I_2$, $I_3$ can be expressed through basic values A, B,

C, depending on main quadratic forms $P_\varepsilon$, $P_\varkappa$, $P_{\varepsilon\varkappa}$ according to the formula (4.18). For this purpose let us note that integral $I_1$ on the basis of (4.6) and (4.16'), (4.17') is expressed directly through function B, after which from (4.11) we find $I_2$ through A and B and then from (4.9) we obtain $I_3$ through A, B, C. In this manner we find the following formulas:

$$I_1=\frac{\sqrt{3}}{2P_\varkappa^{1/2}}B,\quad I_2=\frac{\sqrt{3}P_{\varepsilon\varkappa}}{2P_\varkappa^{3/2}}B+\frac{3}{4P_\varkappa}A,$$
$$I_3=\frac{3\sqrt{3}}{8P_\varkappa^{3/2}}C+\frac{\sqrt{3}P_{\varepsilon\varkappa}^2}{2P_\varkappa^{5/2}}B+\frac{3P_{\varepsilon\varkappa}}{2P_\varkappa^2}A, \qquad (4.19)$$

where to values A, B, C we must either ascribe index "0" and calculate them by the formulas (4.18'), if the flexible strain dominates, or ascribe to the index "1" and calculate according to (4.18") if extension – compression of the middle surface dominates.

An exceptional case, when formula (4.13) and all subsequent calculations lose their meaning, presents the zero-moment stressed state, with which the value $e_i$, and consequently, $\sigma_i$ are constant with respect to thickness. In this case

$$P_\varkappa=P_{\varepsilon\varkappa}=0,\quad e_i=\frac{2}{\sqrt{3}}\sqrt{P_\varepsilon}, \qquad (4.20)$$

and integrals $I_1$, $I_2$, $I_3$ can be calculated directly. From formulas (4.6) we have:

$$I_1=h\frac{\sigma_i}{e_i},\quad I_2=0,\quad I_3=\frac{h^3\sigma_i}{12e_i}, \qquad (4.21)$$

where, inasmuch as equality $P_\varkappa = 0$ is possible only when $\varkappa_1, = \varkappa_2 = = \varkappa_{12} = 0$, then bending moments and torques are equal to zero

$$M_1 = M_2 = M_{12} = 0,$$

and forces are found from simple relationships:

$$T_1-\frac{1}{2}T_2=S_1=h\frac{\sigma_i}{e_i}\varepsilon_1,$$
$$T_2-\frac{1}{2}T_1=S_2=h\frac{\sigma_i}{e_i}\varepsilon_2, \qquad (4.22)$$

$$T_{12} = \frac{2}{3} S_{12} = h \frac{2\sigma_i}{3e_i} \varepsilon_{12}. \qquad (4.22 \text{ cont'd})$$

These relationships coincide with those that take place in plane stressed state, where

$$T_1 = hX_x, \quad T_2 = hY_y, \quad T_{12} = hX_y. \qquad (4.23)$$

Relationships (4.4'), (4.5) or (4.7), (4.8) express forces and moments, acting on the element of the shell, through three quadratic forms (2.6), (2.7), (2.8) $P_\varepsilon$, $P_\varkappa$, $P_{\varepsilon\varkappa}$ and six deformation and distortion components $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_{12}$, $\varkappa_1$, $\varkappa_2$, and $\varkappa_{12}$ and consequently, through three components of the vector of displacement of the middle surface point, inasmuch as deformations and distortions have specific differential expressions through u, v, w.

It is easy to show, that, conversely, all deformations and distortions can be expressed through forces and moments [2].

The set-up relationships, connecting deformations and stresses, take place both in the case of elastoplatic deformations,* and in purely deformations.

Actually, values $\sigma_i$ and $e_i$ are connected with one another by laws,

$$\sigma_i = \Phi(e_i) = 3Ge_i[1 - \omega(e_i)], \qquad (4.24)$$

$$e_i = \Phi^{-1}(\sigma_i) = \frac{\sigma_i}{3G}[1 + \varphi(\sigma_i)], \qquad (4.25)$$

here G is the modulus of elasticity in shear. With respect to curve $\sigma_i = \Phi(e_i)$ we will assume that it satisfies the inequality (Fig. 4)

$$3G \geqslant \frac{\sigma_i}{e_i} > \frac{d\sigma_i}{de_i} > 0. \qquad (4.26)$$

Function $\omega(e_i)$ (function of plasticity by A. A. Il'yushin) constitutes the ratio of line segment MM' to line segment M"M' (Fig. 4). It is equal to zero, as long as deformation is elastic and satisfies the

---

*In the assumption that a simple or close to simple load is realized, i.e., such a load, when all individual loads are proportional to one parameter.

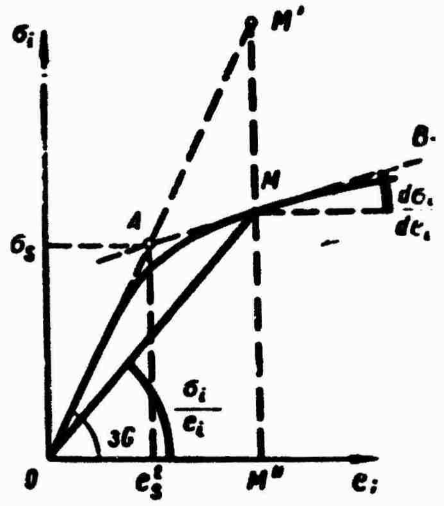


Fig. 4.

following inequality, ensuing from (4.26):

$$1 \geqq \omega + e_i \frac{d\omega}{de_i} > \omega \geqq 0,$$

$$\frac{d\omega}{de_i} > 0. \tag{4.27}$$

The noted properties of functions $\sigma_i$ and $\omega$ correspond to the experiment data. The function of plasticity $\omega$ is expressed though $\sigma_i$ according to equation

$$\omega = \frac{3Ge_i - \sigma_i}{3Ge_i}$$

and analogous to it function $\varphi$ has the expression

$$\varphi = \frac{3Ge_i - \sigma_i}{e_i} = \frac{\omega}{1-\omega}.$$

Designating by $\sigma_s$ and $e_s$ the point, to which the deformation of material may be considered elastic ($\sigma_s$ is the yield point and $e_s$ the flow deformation), we have for function $\omega$:

$$\begin{aligned} &\omega = 0, \quad e_i < e_s, \\ &\omega = \omega(e_i) > 0, \quad e_i > e_s. \end{aligned} \tag{4.28}$$

In the case, when curve $\sigma_i = \Phi(e_i)$ may be replaced with the broken line OAMB , values $\sigma_s$, $e_s$ will correspond to the break point $d\sigma_i/de_i$ will be constant, but for function $\omega$ we obtain:

$$\begin{aligned} &\omega = 0, \quad e_i < e_s; \\ &\omega = \lambda\left(1 - \frac{e_s}{e_i}\right), \quad e_i \geqslant e_s, \end{aligned} \tag{4.29}$$

where constant $\lambda$ (hardening factor) designates the value

$$\lambda = 1 - \frac{1}{3G} \frac{d\sigma_i}{de_i}.$$

Relationship (4.25) does not have any meaning in the case when the shell material was not hardened, i.e., Van Mises' condition of plasticity takes place

$$\sigma_i = \sigma_s$$

or

$$X_x^2 - X_x Y_y + Y_y^2 + 3X_y^2 = \sigma_s^2. \tag{4.30}$$

Now we write the relationships between deformations and stresses in the middle surface for the case of purely elastic deformations:

$$X_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \quad Y_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x),$$
$$X_y = \frac{E}{2(1+\nu)}\gamma, \tag{4.31}$$

where E is the elastic modulus, $\nu$ is Poisson's ratio. Dependences between moments and changes of curvatures will remain the same, as for the plane plate [3]:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$
$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{4.32}$$
$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}.$$

For transverse forces we have expressions:

$$N_x = -D\frac{\partial}{\partial x}\nabla^2 w, \quad N_y = -D\frac{\partial}{\partial y}\nabla^2 w. \tag{4.33}$$

Here D is the cylinder rigidity:

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{4.34}$$

Equations of equilibrium (3.1) and (3.2) are automatically fulfilled upon the introduction of stress function according to formulas:

$$X_x = \frac{\partial^2\tilde{\Phi}}{\partial y^2}, \quad Y_y = \frac{\partial^2\tilde{\Phi}}{\partial x^2}, \quad X_y = -\frac{\partial^2\tilde{\Phi}}{\partial x\,\partial y}, \tag{4.35}$$

where $\tilde{\Phi}$ is the function of stresses in the middle surface, or, in short, the stress function.

We introduce (4.32) and (4.33) in the equation of equilibrium (3.6), then we arrive at the following equation:

$$D\nabla^2\nabla^2 w = X_x h\left(k_x + \frac{\partial^2 w}{\partial x^2}\right) + Y_y h\left(k_y + \frac{\partial^2 w}{\partial y^2}\right) -$$
$$-2X_y h\frac{\partial^2 w}{\partial x\,\partial y} + q. \quad (4.36)$$

Let us now transform the condition of deformation compatibility (2.10). Expressing deformations $\varepsilon_x$, $\varepsilon_y$, $\gamma$ through stresses, we find:

$$\frac{\partial^2 X_x}{\partial y^2} - 2\frac{\partial^2 X_y}{\partial x\,\partial y} + \frac{\partial^2 Y_y}{\partial x^2} - \nu\left(\frac{\partial^2 X_x}{\partial y^2} + 2\frac{\partial^2 X_y}{\partial x\,\partial y} + \frac{\partial^2 Y_y}{\partial x^2}\right) =$$
$$= E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - k_x\frac{\partial^2 w}{\partial y^2} - k_y\frac{\partial^2 w}{\partial x^2}\right]. \quad (4.37)$$

We introduce in equations (4.36) and (4.37) the stress function $\tilde{\Phi}$ according to (4.35). Then basic equations of the theory of flexible sloping shells will take the following form,*

$$\frac{D}{h}\nabla^2\nabla^2 w = L(w, \tilde{\Phi}) + k_x\frac{\partial^2\tilde{\Phi}}{\partial y^2} + k_y\frac{\partial^2\tilde{\Phi}}{\partial x^2} + \frac{q}{h}, \quad (4.38)$$
$$\frac{1}{E}\nabla^2\nabla^2\tilde{\Phi} = -\frac{1}{2}L(w, w) - k_x\frac{\partial^2 w}{\partial y^2} - k_y\frac{\partial^2 w}{\partial x^2}. \quad (4.39)$$

Here through $\nabla^2\nabla^2(\quad)$ Laplacian operator is designated.

$$\nabla^2\nabla^2(\quad) = \frac{\partial^4(\ )}{\partial x^4} + 2\frac{\partial^4(\ )}{\partial x^2\,\partial y^2} + \frac{\partial^4(\ )}{\partial y^4}.$$

In the particular case of circular cylindrical shell with the radius R we obtain

$$\frac{D}{h}\nabla^2\nabla^2 w = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2\tilde{\Phi}}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2\tilde{\Phi}}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2\tilde{\Phi}}{\partial x\,\partial y} +$$
$$+\frac{1}{R}\frac{\partial^2\tilde{\Phi}}{\partial x^2} + \frac{q}{h}; \quad (4.40)$$
$$\frac{1}{E}\nabla^2\nabla^2\tilde{\Phi} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}. \quad (4.41)$$

In examining only small sags of the shell we can disregard nonlinear terms in equations (4.38) and (4.39). Here we obtain:

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*These equations are applicable also in examining of shells of an arbitrary outline, if deformation has the character of local loss of stability [4].

$$\frac{D}{h}\nabla^2\nabla^2 w = k_x \frac{\partial^2\tilde{\Phi}}{\partial y^2} + k_y \frac{\partial^2\tilde{\Phi}}{\partial x^2} + \frac{q}{h}; \tag{4.42}$$

$$\frac{1}{E}\nabla^2\nabla^2\tilde{\Phi} = -k_x \frac{\partial^2 w}{\partial y^2} - k_y \frac{\partial^2 w}{\partial x^2}. \tag{4.43}$$

Considering that curvature $k_x$ and $k_y$ are constants and applying to (4.42) operator $\nabla^2\ \nabla^2$, we find:

$$\frac{D}{h}\nabla^2\nabla^2\nabla^2\nabla^2 w = k_x \frac{\partial^2}{\partial y^2}(\nabla^2\nabla^2\tilde{\Phi}) + k_y \frac{\partial^2}{\partial x^2}(\nabla^2\nabla^2\tilde{\Phi}) + \frac{1}{h}\nabla^2\nabla^2 q.$$

If we substitute here $\nabla^2\ \nabla^2\tilde{\Phi}$ according to (4.43), then it is possible to reduce system (4.42)-(4.43) to one solving equation of the eighth order with respect to function w:

$$\frac{D}{Eh}\nabla^2\nabla^2\nabla^2\nabla^2 w + k_x^2 \frac{\partial^4 w}{\partial y^4} + 2k_x k_y \frac{\partial^4 w}{\partial x^2 \partial y^2} + k_y^2 \frac{\partial^4 w}{\partial x^4} = \frac{1}{Eh}\nabla^2\nabla^2 q. \tag{4.44}$$

For the circular cylindrical shell the equation will have the form

$$\frac{D}{Eh}\nabla^2\nabla^2\nabla^2\nabla^2 w + \frac{1}{R^2}\frac{\partial^4 w}{\partial x^4} = \frac{1}{Eh}\nabla^2\nabla^2 q. \tag{4.45}$$

Let us now investigate the case, when we must take into consideration initial forces in the middle surfaces, constant in value, which is necessary, for instance, in problems of stability of shells,

$$\frac{\partial^2\tilde{\Phi}_{\text{нч}}}{\partial y^2} = -P_x, \quad \frac{\partial^2\tilde{\Phi}_{\text{нч}}}{\partial x^2} = -P_y, \quad \frac{\partial^2\tilde{\Phi}_{\text{нч}}}{\partial x\,\partial y} = S. \tag{4.46}$$

Then equation (4.40) will assume the form (when $q \equiv 0$)

$$\begin{aligned}\frac{D}{h}\nabla^2\nabla^2 w = &-P_x \frac{\partial^2 w}{\partial x^2} - P_y \frac{\partial^2 w}{\partial y^2} - 2S\frac{\partial^2 w}{\partial x\,\partial y} + \\ &+ k_x \frac{\partial^2\tilde{\Phi}}{\partial y^2} + k_y \frac{\partial^2\tilde{\Phi}}{\partial x^2}.\end{aligned} \tag{4.47}$$

Taking into consideration (4.43), we obtain the solving equation

$$\begin{aligned}&\frac{D}{h}\nabla^2\nabla^2\nabla^2\nabla^2 w + k_x^2 \frac{\partial^4 w}{\partial y^4} + 2k_x k_y \frac{\partial^4 w}{\partial x^2\,\partial y^2} + k_y^2 \frac{\partial^4 w}{\partial x^4} + \\ &+ \frac{P_x}{E}\nabla^2\nabla^2\left(\frac{\partial^2 w}{\partial x^2}\right) + \frac{2s}{E}\nabla^2\nabla^2\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) + \frac{P_y}{E}\nabla^2\nabla^2\left(\frac{\partial^2 w}{\partial y^2}\right) = 0.\end{aligned} \tag{4.48}$$

In the case of circular cylindrical shell:

$$\frac{D}{h}\nabla^2\nabla^2\nabla^2\nabla^2 w + \frac{E}{R^2}\frac{\partial^4 w}{\partial x^4} + P_x\nabla^2\nabla^2\left(\frac{\partial^2 w}{\partial x^2}\right) + 2S\nabla^2\nabla^2\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) + P_y\nabla^2\nabla^2\left(\frac{\partial^2 w}{\partial y^2}\right) = 0. \quad (4.49)$$

Assuming that in (4.36) $w \equiv 0$, we obtain the equation of theory of zero-moment shells:

$$k_x X_x + k_y Y_y = -\frac{q}{h};$$

let us note that the transverse load is considered positive in the direction toward the center of the curvature.

## § 5. Variational Formulation of the Problem of the Theory of Shells With Geometric Nonlinearity Taken Into Account

Let us examine geometric nonlinearity* of a shell, being in equilibrium under action of volume $\overline{F}$ and surface $\overline{P}_n$ forces. Let us assume that $\delta u_1^z$, $\delta u_2^z$, $\delta w^z$ – infinitesimal variations of displacements. Work of external forces on variations of displacements will be

$$\delta A = \iiint_{\Omega} \overline{F}\delta\overline{u}\,d\Omega + \iint_{\Sigma} \overline{P}_n\delta\overline{u}\,d\Sigma, \quad (5.1)$$

where $d\Omega = A^*B^*d\alpha d\beta dz$ is the element of the volume of shell, $\Sigma$ is the general boundary surface of the shell; $\alpha$ and $\beta$ are curvilinear orthogonal coordinates, determining the position of the point on the middle surface before and after deformation, asterisks mark the values, pertaining to the deformed shell or its middle surface, $\delta\overline{u}$ is the variation displacement, equal to

$$\delta\overline{u} = \delta\overline{v} + z\delta\overline{n}^*. \quad (5.2)$$

*Deformations are expressed as derivatives of displacements taking into account nonlinear terms. For instance, $e_x = \partial u/\partial x + 1/2(\partial w/\partial x)^2 - k_1 w$.

Here $\delta\bar{v}$ is the variation of vector of displacement of points of the middle surface; $\delta\bar{n}^*$ is the variation of the vector of the normal to the middle surface. The stress vector $\bar{P}_n$ on the area with the normal $\bar{n}$ is expressed through stress vectors $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_z$, acting on the areas taken on coordinate surfaces $\alpha$ = const, $\beta$ = const and $z$ = const, according to the known formula of the theory of elasticity

$$\bar{P}_n = \bar{P}_1 \cos(n\alpha) + \bar{P}_2 \cos(n\beta) + \bar{P}_z \cos(nz).$$

Let us set up the expression of the virtual work of external forces $\delta A$ through the energy of deformation of the shell. Putting expression $\bar{P}_n$ in (5.1), we obtain

$$\delta A = \iiint\limits_{\Omega} \bar{F}\delta\bar{u}\, d\Omega + \iint\limits_{\Sigma} \{\bar{P}_1 \cos(n\alpha) + \bar{P}_2 \cos(n\beta) +$$

$$+ \bar{P}_z \cos(nz)\}\, \delta\bar{u}\, d\Sigma.$$

Hence, using formula of transformation of the surface integral into the volume integral and taking into account equations of equilibrium, we find:

$$\delta A = \iiint\limits_{\Omega} \left\{ \bar{P}_1 B^* (\delta\bar{u})_{,1} + \bar{P}_2 A^* (\delta\bar{u})_{,2} + \bar{P}_z A^* B^* (\delta\bar{u})_{,z} \right\} d\alpha\, d\beta\, dz, \qquad (5.3)$$

where $(\delta\bar{u})_{,i} = (\delta\bar{v})_{,i} + z\delta\bar{n}^*_{,i} = \delta\bar{r}^*_{,i} + z\delta\bar{n}^*_{,i}$, $(\delta\bar{u})_{,z} = \delta\bar{n}^*$, where "," below, before the index designates differentiation by $\alpha$ or $\beta$ once. This is the expression of the principle of virtual displacements for the shell, considered as a three-dimensional body. Formula (5.3) may be also written in this form:

$$\delta A = \iint\limits_{\Sigma} \delta W\, d\Sigma, \qquad (d\Sigma = A^* B^*\, d\alpha\, d\beta), \qquad (5.4)$$

where

$$\delta W = \sum_{ij} \{ T^*_{ij}\delta\tilde{\varepsilon}_{ji} + M^*_{ij} (\delta\varkappa_{ij} - \sum_{s=1}^{2} k^*_{is}\delta\tilde{\varepsilon}_{sj}) \}, \qquad (5.5)$$

this is the variation of the deformation energy of the shell, referred to one unit of area of the middle surface. If we use the simplest variant of elasticity relationships — the Hooke's law, — and reject

values of the order of $k_{is}\varepsilon_{ik}$ ($k_{is}$ are curvatures of coordinate lines $\alpha$ and $\beta$, for instance, with small deformations $k_{11}^* \simeq 1/R_{\alpha\beta}^*$) and take into consideration the equality $T_{ij}^* = T_{ji}^*$ (here $\delta\tilde{\varepsilon}_{ii} = \delta\varepsilon_{ii}$, $\delta\tilde{\varepsilon}_{12} + \delta\tilde{\varepsilon}_{21} = 2\delta\varepsilon_{12}$), then (5.5) takes this form:

$$\delta W = \sum_{ij} (T_{ij}^{\bullet}\delta\varepsilon_{ij} + M_{ij}^{\bullet}\delta\varkappa_{ij}), \tag{5.6}$$

where $\varepsilon_{ij}$ and $\varkappa_{ij}$ are expressed by formulas:

$$2\varepsilon_{ij} = e_{ij} + e_{ji} + \sum_{s=1}^{2} e_{is}e_{js} + \omega_i\omega_j \qquad (i,\ j = 1,\ 2), \tag{5.7}$$

$$\varkappa_{11} = k_{11}e_{22} - k_{12}e_{21} - \frac{1}{A}\left(E_1\frac{\partial e_{11}}{\partial\alpha} + E_2\frac{\partial e_{12}}{\partial\alpha} + E_3\frac{\partial\omega_1}{\partial\alpha}\right) - \frac{\omega_2}{AB}\frac{\partial A}{\partial\beta}, \tag{5.8}$$

$$\varkappa_{12} = k_{12}e_{11} - k_{11}e_{21} - \frac{1}{A}\left(E_1\frac{\partial e_{21}}{\partial\alpha} + E_2\frac{\partial e_{22}}{\partial\alpha} + E_3\frac{\partial\omega_2}{\partial\alpha}\right) + \frac{\omega_1}{AB}\frac{\partial B}{\partial\beta}. \quad (\overrightarrow{\underleftarrow{1,2}})$$

Parameters $e_{ij}$, $\omega_i$, $E_i$, $E_3$ characterize angles of rotation of coordinate vectors $\bar{r}_{,i}$, $\bar{n}$ in the process of deformation, for instance $\bar{n}^* = \bar{e}_1E_1 + \bar{e}_2E_2 + \bar{n}E_3$, here $E_i = e_{i1}\omega_1 + e_{i2}\omega_2 - (1 + e_{11} + e_{22})\omega_i$, $E_3 = (1 + e_{11})(1 + e_{22}) - e_{12}e_{21}$.

Thus, the variation of strain energy of the shell is composed of variation of the stretch and shear energies:

$$\delta W_1 = T_{11}^{\bullet}\delta\varepsilon_{11} + T_{22}^{\bullet}\delta\varepsilon_{22} + 2T_{12}^{\bullet}\delta\varepsilon_{12} \tag{5.9}$$

and the variation of the bending and torsion energies:

$$\delta W_2 = M_{11}^{\bullet}\delta\varkappa_{11} + M_{22}^{\bullet}\delta\varkappa_{22} + 2M_{12}^{\bullet}\delta\varkappa_{12}. \tag{5.10}$$

Putting in (5.6) forces and moments,

$$T_{11}^{\bullet} = K(\varepsilon_{11} + \nu\varepsilon_{22}),\quad T_{12}^{\bullet} = T_{21}^{\bullet} = K(1-\nu)\varepsilon_{12},\quad T_{22}^{\bullet} = K(\varepsilon_{22} + \nu\varepsilon_{11}),$$
$$M_{11}^{\bullet} = D(\varkappa_{11} + \nu\varkappa_{22}),\quad M_{12}^{\bullet} = M_{21}^{\bullet} = D(1-\nu)\varkappa_{12},\quad M_{22}^{\bullet} = D(\varkappa_{22} + \nu\varkappa_{11})$$

(here $K = Eh/1 - \nu^2$ is the extension-compression rigidity and $D = Eh^3/12(1 - \nu^2)$ is the cylinder rigidity) and integrating by

deformation components in the range from zero state to the state with deformations $\varepsilon_{ik}$ and $\varkappa_{ik}$, we find the expression of the specific work of deformation of the shell:

$$2W = K[(\varepsilon_{11}+\varepsilon_{22})^2 - 2(1-\nu)(\varepsilon_{11}\varepsilon_{22}-\varepsilon_{12}^2)] + \\ + D[(\varkappa_{11}+\varkappa_{22})^2 - 2(1-\nu)(\varkappa_{11}\varkappa_{22}-\varkappa_{12}^2)]. \tag{5.11}$$

where $\varepsilon_{ik}$ and $\varkappa_{ik}$ are expressed through displacements $u_1$, $u_2$, w according to formulas (5.7) and (5.8). Formula (5.11) is analogous to the formula of deformation energy of the plate. Applying it to the thin shell, we allow an error of the order of h/R as compared to 1.

Now, after a number of simple, but sufficiently labor-consuming transformations, we find the expression for variation of the work of all possible external forces, which may be recorded thus:

$$\delta A = \iint_{\Sigma} (\overline{X}\delta\overline{v} + [\overline{L}\times\overline{n}^*]\delta\overline{n}^*)\,d\Sigma + \\ + \int_{c} (\overline{\Phi}\delta\overline{v} + \widetilde{G}^*\overline{m}^*\delta\overline{n}^*)\,ds + \widetilde{H}^*\overline{n}^*\delta\overline{v}|_c, \tag{5.12}$$

where $\overline{\Phi}$ is the vector of external boundary force, $\widetilde{G}^*$ and $\widetilde{H}^*$ are the external bending moment and torque on the contour of deformed shell.

The surface integral in (5.12) is the work of external forces on infinitesimal variations of displacements and external moments on infinitesimal variations of angles of turn, since

$$[\overline{L}\times\overline{n}^*]\delta\overline{n}^* = \sum_{i=1}^{2} L_i\overline{e}_i^*\delta\overline{n}^*, \tag{5.13}$$

where $\overline{e}_i^*\delta\overline{n}^*$ are variations of angles of turn.

The contour integral in (5.12) is the work of external forces and moments on variations of displacements and angle of rotation, since $\overline{m}^*\overline{\delta n}^*$ is the variation of angle of rotation around the tangent to the contour.

Outside-the-integral term $\widetilde{H}^*\overline{n}^*\delta\overline{v}|_c$ is the work of concentrated boundary forces on displacements. It disappears, if shell edges are

either supported on hinges or rigidly fastened. It disappears also when the contour does not have any angle points, and neither $\widetilde{H}^*$ nor $\overline{v}$ can have discontinuities. If the shell contour contains angular points, concentrated forces of the $\widetilde{H}^*\overline{n}^*$ type can appear in angles in the form of reactive concentrated forces. Thus, the variational equation of the principle of possible displacements in the nonlinear theory of shells will be expressed by the relationship

$$\delta A = \iint_{\Sigma} \delta W AB\, d\alpha\, d\beta, \tag{5.14}$$

where $\delta W$ is yielded either by formula (5.5) or (5.6) and $\delta A$ by formula (5.12). Let us note that the variational equation of the type (5.14) is also true for the general nonlinear theory of shells, where displacements and deformations considered arbitrary.

Variational equation (5.14) may be interpreted in the following manner. Let us assume that $Э_1$ is the potential strain energy of the shell, $\delta Э_1$ is its full variation in isothermal or adiabatic deformation processes:

$$Э_1 = \iint_{\Sigma} W\, d\Sigma, \quad \delta Э_1 = \iint_{\Sigma} \delta W\, d\Sigma.$$

Let us further assume that $\delta Э_2 = -\delta A$ is the variation of potential load energy. Then it is possible to record (5.14) in the form

$$\delta Э = \delta Э_1 + \delta Э_2, \tag{5.15}$$

where $Э$ is the full potential energy of the system.

Thus, the state of equilibrium of the shell differs from adjacent geometrically possible states by the fact that with any infinitesimal virtual displacements of the system from the position of equilibrium the increase of full potential energies is equal to zero. This is Lagrange's variational principle. Geometrically possible states of shell are such, states, with which displacement variations do not disturb holonomic constraints, superimposed on the shell. Geometric

boundary conditions, and also comparable in the Lagrange variational principle values $\varepsilon_{ik}$ and $\varkappa_{ik}$, which have to present continuous deformations, satisfying conditions of deformation continuity can serve as holonomic constraints. This condition will be assured, if deformations $\varepsilon_{ik}$ and $\varkappa_{ik}$ are expressed through displacements $u_i$ and w through formulas (5.7) and (5.8).

An increase of the work of external forces and moments $\delta A$ is a full variation only in certain particular cases: for instance, when external forces can be considered to be independent of deformations and, furthermore, parameters $e_{ik}$ are small, i.e., $e_{ik} \sim \varepsilon_p$ ($\varepsilon_p$ is the elongation per unit length at the proportionality limit of the shell material.

Let us rewrite the variational equation (5.15) in the form

$$\delta(\partial_1 + \partial_2) = \delta\partial = 0. \tag{5.16}$$

This equation is also true for end sags under the condition that edges of the shell are either supported on hinges or rigidly fastened and, furthermore, external forces tolerate the potential

$$X_1 = \frac{\partial f}{\partial u_1}, \quad X_2 = \frac{\partial f}{\partial u_2}, \quad X_3 = \frac{\partial f}{\partial w}, \quad (M_i = 0).$$

Consequently, (5.16) can be formulated in this manner: <u>from all virtual displacements, congruent with holonomic constraints, superimposed on the shell, in reality only those take place for which the potential energy of system $\partial$ assumes the steady-state value,</u> i.e., $\delta\partial = 0$.

From the variational equation (5.14) equations of equilibrium and static boundary conditions ensue.

Let us note that earlier we obtained fundamental equations of the shell theory proceeding from the principle of virtual displacements.

On the variational principle of virtual displacements the Ritz approximation method (energy method) is based, the essence of which consists in the following. Variational equation (5.14) contains the equation of equilibrium and static boundary conditions. Therefore, satisfying this variational equation, we thereby satisfy static conditions inside the shell and on the contour. The latter conditions will be executed in the process of resolution of the problem with the exactness that will be the greater the higher approximation of the problem's solution. Moreover, geometric boundary conditions are essential, i.e., they have to be satisfied beforehand. Therefore, in the approximate solution of specific problems with the help of variational equation (5.14) we shall prescribe approximating functions of the form:

$$u_1 = \sum_{k=1}^{n} A_k f_k(\alpha, \beta), \quad u_2 = \sum_{k=1}^{n} B_k \varphi_k(\alpha, \beta), \quad w = \sum_{k=1}^{n} C_k \psi_k(\alpha, \beta), \tag{5.17}$$

where $A_k$, $B_k$, $C_k$ are constants to be determined , and $f_k$, $\varphi_k$, $\psi_k$ are given functions, which are chosen in such a manner that the displacements $u_1$, $u_2$, $w$, permissible in comparisons, satisfy geometric boundary conditions. Then, putting (5.17) in (5.14) and comparing factors in variations $\delta A_k$, $\delta B_k$, $\delta C_k$, we obtain the system of algebraic equations for determination of constants $A_k$, $B_k$, $C_k$ sought. In the general instance the system of obtained algebraic equations will be nonlinear. It will be linear only in linear problems of the theory of shells. In concrete instances along with difficulties of selection of approximating functions (5.17) the resolution of the obtained nonlinear system presents difficulties of a purely algebraic character. But, in spite of this the Ritz method is the most widely used and reliable method.

## § 6. Improved Motion Equations in Moments and Forces*

Let us assume that $\sigma_{ik}$ is the projection of stress on the area, the normal to which in undeformed state coincided with the direction of coordinate line i, on the direction, which before deformation coincided with the direction of coordinate line k. Let us introduce forces and moments, with usual formulas

$$T_{ik} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ik}\,dz, \quad M_{ik} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ik} z\,dz, \quad Q_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{iz}\,dz \qquad (i,\ k = 1,\ 2). \tag{6.1}$$

Formulas, by means of which forces, moments and intercepting forces are expressed through deformation components, will coincide with known formulas for the shell with small sags and with formulas for the plate with large sags.

Let us now write motion equations of the theory of elasticity in curvilinear coordinates in projections on directions 1, 2, 3 in undeformed body,

$$\begin{aligned}
&\frac{\partial}{\partial \alpha_1}(H_2H_3S_{11}) + \frac{\partial}{\partial \alpha_2}(H_3H_1S_{21}) + \frac{\partial}{\partial \alpha_3}(H_1H_2S_{31}) + H_3\frac{\partial H_1}{\partial \alpha_2}S_{12} + \\
&\quad + H_2\frac{\partial H_1}{\partial \alpha_3}S_{13} - H_3\frac{\partial H_2}{\partial \alpha_1}S_{22} - H_2\frac{\partial H_3}{\partial \alpha_1}S_{33} = \rho H_1H_2H_3\frac{\partial^2 \tilde{u}}{\partial t^2}, \\
&\frac{\partial}{\partial \alpha_1}(H_2H_3S_{12}) + \frac{\partial}{\partial \alpha_2}(H_3H_1S_{22}) + \frac{\partial}{\partial \alpha_3}(H_1H_2S_{32}) + H_1\frac{\partial H_2}{\partial \alpha_3}S_{23} + \\
&\quad + H_3\frac{\partial H_2}{\partial \alpha_1}S_{21} - H_1\frac{\partial H_3}{\partial \alpha_2}S_{33} - H_3\frac{\partial H_1}{\partial \alpha_2}S_{11} = \rho H_1H_2H_3\frac{\partial^2 \tilde{v}}{\partial t^2}, \\
&\frac{\partial}{\partial \alpha_1}(H_2H_3S_{13}) + \frac{\partial}{\partial \alpha_2}(H_3H_1S_{23}) + \frac{\partial}{\partial \alpha_3}(H_1H_2S_{33}) + H_2\frac{\partial H_3}{\partial \alpha_1}S_{31} + \\
&\quad + H_1\frac{\partial H_3}{\partial \alpha_2}S_{32} - H_2\frac{\partial H_1}{\partial \alpha_3}S_{11} - H_1\frac{\partial H_2}{\partial \alpha_3}S_{22} = \rho H_1H_2H_3\frac{\partial^2 \tilde{w}}{\partial t^2}.
\end{aligned} \tag{6.2}$$

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*The material presented here and in §§ 7 and 9 was kindly offered us by the senior scientific colleague of the Institute of Mechanics of the Academy of Sciences of USSR, Candidate of Physical and Mathematical Sciences, M. P. Galin [5].

The reader will find a survey of the contemporary state of the statics theory of thin shells in E. Reysner's article "Certain problems of the shell theory. Elastic shells, Foreign literature (IL), 1962.

Here the following designations are introduced

$$
\begin{aligned}
S_{11} &= (1+e_{11})\sigma^*_{11} + \left(\frac{1}{2}e_{12} - \omega_3\right)\sigma^*_{12} + \left(\frac{1}{2}e_{13} + \omega_2\right)\sigma^*_{13},\\
S_{12} &= \left(\frac{1}{2}e_{12} + \omega_3\right)\sigma^*_{11} + (1+e_{22})\sigma^*_{12} + \left(\frac{1}{2}e_{23} - \omega_1\right)\sigma^*_{13},\\
S_{13} &= \left(\frac{1}{2}e_{13} - \omega_2\right)\sigma^*_{11} + \left(\frac{1}{2}e_{23} + \omega_1\right)\sigma^*_{12} + (1+e_{33})\sigma^*_{13},\\
S_{21} &= (1+e_{11})\sigma^*_{21} + \left(\frac{1}{2}e_{12} - \omega_3\right)\sigma^*_{22} + \left(\frac{1}{2}e_{13} + \omega_2\right)\sigma^*_{23},\\
S_{22} &= \left(\frac{1}{2}e_{12} + \omega_3\right)\sigma^*_{21} + (1+e_{22})\sigma^*_{22} + \left(\frac{1}{2}e_{23} - \omega_1\right)\sigma^*_{23},\\
S_{23} &= \left(\frac{1}{2}e_{13} - \omega_2\right)\sigma^*_{21} + \left(\frac{1}{2}e_{23} + \omega_1\right)\sigma^*_{22} + (1+e_{33})\sigma^*_{23},\\
S_{31} &= (1+e_{11})\sigma^*_{31} + \left(\frac{1}{2}e_{12} - \omega_3\right)\sigma^*_{32} + \left(\frac{1}{2}e_{13} + \omega_2\right)\sigma^*_{33},\\
S_{32} &= \left(\frac{1}{2}e_{12} + \omega_3\right)\sigma^*_{31} + (1+e_{22})\sigma^*_{32} + \left(\frac{1}{2}e_{23} - \omega_1\right)\sigma^*_{33},\\
S_{33} &= \left(\frac{1}{2}e_{13} - \omega_2\right)\sigma^*_{31} + \left(\frac{1}{2}e_{23} + \omega_1\right)\sigma^*_{32} + (1+e_{33})\sigma^*_{33}.
\end{aligned}
\tag{6.3}
$$

Values

$$\sigma^*_{ij} = \frac{S^*_i}{S_i}\frac{\sigma_{ij}}{1+E_j}, \quad \sigma^*_{ij} = \sigma^*_{ji}$$

constitute stresses, referred to initial dimensions of the element, the dimensions of faces of which are increased by $E_j$ and their area becomes equal to $S^*_i$ instead of $S_i$, aand here

$$
\begin{gathered}
E_j = \sqrt{2\varepsilon_{jj}+1} - 1,\\
\frac{S^*_i}{S_i} = \sqrt{(1+2\varepsilon_{jj})(1+2\varepsilon_{kk}) - \varepsilon^2_{jk}} \qquad (i,\ j,\ k = 1,\ 2,\ 3).
\end{gathered}
\tag{6.4}
$$

With small deformations $1 + e_{ii} \approx 1$, $\sigma^*_{ij} \approx \sigma_{ij}$. Subsequently we shall consider deformation to be small and even in initial equations will assume that $\sigma^*_{ij} = \sigma_{ij}$. Strictly speaking, in differentiation of values $\sigma_{ij}$ we should bear in mind that

$$
\begin{aligned}
\frac{\partial\sigma^*_{ij}}{\partial\alpha_r} = \frac{S^*_i}{S_i}\frac{1}{1+E_j}\frac{\partial\sigma_{ij}}{\partial\alpha_r} + \sigma_{ij}\Bigg[\frac{\partial}{\partial\alpha_r}\left(\frac{S^*_i}{S_i}\right)\frac{1}{1+E_j} - {}\\
-\left(\frac{S^*_i}{S_i}\right)\frac{1}{(1+E_j)^2}\frac{\partial E_j}{\partial\alpha_r}\Bigg] \qquad (r = 1,\ 2,\ 3).
\end{aligned}
$$

since

$$\frac{\partial E_j}{\partial\alpha_r} = \frac{1}{\sqrt{1+2\varepsilon_{jj}}}\frac{\partial\varepsilon_{jj}}{\partial\alpha_r},$$

$$\frac{\partial}{\partial \alpha_r}\left(\frac{S_i^{\bullet}}{S_i}\right)=\frac{1}{\sqrt{(1+2\varepsilon_{jj})(1+2\varepsilon_{kk})-\varepsilon_{jk}^2}}\left[(1+2\varepsilon_{kk})\frac{\partial \varepsilon_{jj}}{\partial \alpha_r}+\right.$$
$$\left.+(1+2\varepsilon_{jj})\frac{\partial \varepsilon_{kk}}{\partial \alpha_r}-\varepsilon_{jk}\frac{\partial \varepsilon_{jk}}{\partial \alpha_r}\right],$$

then in small deformations

$$\frac{\partial \sigma_{ij}^{\bullet}}{\partial \alpha_r}=\frac{\partial \sigma_{ij}}{\partial \alpha_r}+\sigma_{ij}\left(\frac{\partial \varepsilon_{kk}}{\partial \alpha_r}-\varepsilon_{jk}\frac{\partial \varepsilon_{jk}}{\partial \alpha_r}\right)\approx\frac{\partial \sigma_{ij}}{\partial \alpha_r}+\sigma_{ij}\frac{\partial \varepsilon_{kk}}{\partial \alpha_r}.$$

In accordance with the expressions accepted for displacements $\partial \varepsilon_{zz}/\partial \alpha_r = 0$, therefore

$$\frac{\partial \sigma_{ij}^{\bullet}}{\partial \alpha_r}=\frac{\partial \sigma_{ij}}{\partial \alpha_r},\quad \frac{\partial \sigma_{iz}^{\bullet}}{\partial \alpha_r}=\frac{\partial \sigma_{iz}}{\partial \alpha_r}+\sigma_{iz}\frac{\partial \varepsilon_{jj}}{\partial \alpha_r}\qquad (i,\ j=1,\ 2). \tag{6.5}$$

Taking into consideration that $\sigma_{zz} = 0$ when $-h/2 < z < h/2$, we shall also have

$$\frac{\partial \sigma_{zz}^{\bullet}}{\partial \alpha_i}=0\qquad (i=1,\ 2).$$

Let us assume that the shell is under action of external and internal pressures $P_n(h/2)$ and $P_n(-h/2)$, and also of external and internal tangent of loads $P_{s1}(h/2)$, $P_{s2}(h/2)$ and $P_{s1}(-h/2)$, $P_{s2}(-h/2)$. In the case on surfaces $z = \pm h/2$ the following conditions must be fulfilled:

$$\begin{aligned}
&\sigma_{zz}\left(\frac{h}{2}\right)=-P_n\left(\frac{h}{2}\right),\quad \sigma_{zz}\left(-\frac{h}{2}\right)=-P_n\left(-\frac{h}{2}\right),\\
&\sigma_{z1}\left(\frac{h}{2}\right)=-P_{s1}\left(\frac{h}{2}\right),\quad \sigma_{z1}\left(-\frac{h}{2}\right)=-P_{s1}\left(-\frac{h}{2}\right),\\
&\sigma_{z2}\left(\frac{h}{2}\right)=-P_{s2}\left(\frac{h}{2}\right),\quad \sigma_{z2}\left(-\frac{h}{2}\right)=-P_{s2}\left(-\frac{h}{2}\right).
\end{aligned} \tag{6.6}$$

We shall first integrate motion equations by z from -h/2 to h/2, and then multiply the first two equations by z and also integrate them within the same limits; then, disregarding the effect of moments of stress on the conditions of equilibrium of forces and the effect of moments of tangential stresses and moments of the highest order

on conditions of equilibrium of moments, after simple transformations we will obtain:

$$
\begin{gathered}
\frac{\partial}{\partial \alpha_1}(BT_{11})+\frac{\partial}{\partial \alpha_1}\left[B\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)T_{12}\right]-\frac{\partial}{\partial \alpha_1}(\varphi BQ_1)+\\
+\frac{\partial}{\partial \alpha_2}(AT_{21})+\frac{\partial}{\partial \alpha_2}\left[A\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)T_{22}\right]-\frac{\partial}{\partial \alpha_2}(\varphi AQ_2)+\\
+\frac{\partial A}{\partial \alpha_2}\left[\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)T_{11}+T_{12}-(\psi Q_1)\right]-\\
-\frac{\partial B}{\partial \alpha_1}\left[\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)T_{21}+T_{22}-(\psi Q_2)\right]+\\
+\frac{AB}{R_1}\left[\left(\frac{1}{A}\frac{\partial w}{\partial \alpha_1}-\frac{u}{R_1}\right)T_{11}+\left(\frac{1}{B}\frac{\partial w}{\partial \alpha_2}-\frac{v}{R_2}\right)T_{12}+Q_1\right]+\\
+AB\left\{\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]+\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)\times\right.\\
\times\left[P_{s2}\left(\frac{h}{2}\right)+P_{s2}\left(-\frac{h}{2}\right)-\frac{h}{2}\left(\frac{1}{B}\frac{\partial \varphi}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}\psi\right)\times\right.\\
\left.\times\left[P_{s2}\left(\frac{h}{2}\right)-P_{s2}\left(-\frac{h}{2}\right)\right]-\varphi\left[-P_n\left(\frac{h}{2}\right)+P_n\left(-\frac{h}{2}\right)\right]\right\}-\\
-\varphi\Delta=\rho hAB\frac{\partial^2 u}{\partial t^2},
\end{gathered} \tag{6.7}
$$

$$
\begin{gathered}
\frac{\partial}{\partial \alpha_2}(AT_{22})+\frac{\partial}{\partial \alpha_2}\left[A\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)T_{21}\right]-\\
-\frac{\partial}{\partial \alpha_2}(\psi AQ_2)+\frac{\partial}{\partial \alpha_1}(BT_{12})+\frac{\partial}{\partial \alpha_1}\left[B\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)T_{11}\right]-\\
-\frac{\partial}{\partial \alpha_{11}}(\psi BQ_1)+\frac{\partial}{\partial \alpha_1}\left[\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)T_{22}+T_{21}-(\varphi Q_2)\right]-\\
-\frac{\partial A}{\partial \alpha_2}\left[\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)T_{12}+T_{11}-(\varphi Q_1)\right]+\\
+\frac{AB}{R_2}\left[\left(\frac{1}{B}\frac{\partial w}{\partial \alpha_2}-\frac{v}{R_2}\right)T_{22}+\left(\frac{1}{A}\frac{\partial w}{\partial \alpha_1}-\frac{u}{R_1}\right)T_{21}+Q_2\right]+\\
+AB\left\{\left[P_{s2}\left(\frac{h}{2}\right)+P_{s2}\left(-\frac{h}{2}\right)\right]+\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)\times\right.\\
\times\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]-\frac{h}{2}\left(\frac{1}{A}\frac{\partial \psi}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}\varphi\right)\times\\
\left.\times\left[P_{s1}\left(\frac{h}{2}\right)-P_{s1}\left(-\frac{h}{2}\right)\right]-\psi\left[-P_n\left(\frac{h}{2}\right)+P_n\left(-\frac{h}{2}\right)\right]\right\}-\\
-\psi\Delta=\rho hAB\frac{\partial^2 v}{\partial t^2},
\end{gathered}
$$

$$
\begin{gathered}
\frac{\partial}{\partial\alpha_1}(BQ_1)+\frac{\partial}{\partial\alpha_1}\left[B\left(\frac{1}{A}\frac{\partial w}{\partial\alpha_1}-\frac{u}{R_1}\right)T_{11}\right]+\\
+\frac{\partial}{\partial\alpha_1}\left[B\left(\frac{1}{B}\frac{\partial w}{\partial\alpha_2}-\frac{v}{R_2}\right)T_{12}\right]+\frac{\partial}{\partial\alpha_2}(AQ_2)+\\
+\frac{\partial}{\partial\alpha_2}\left[A\left(\frac{1}{B}\frac{\partial w}{\partial\alpha_2}-\frac{v}{R_2}\right)T_{22}\right]+\frac{\partial}{\partial\alpha_2}\left[A\left(\frac{1}{A}\frac{\partial w}{\partial\alpha_1}-\frac{u}{R_1}\right)T_{21}\right]-\\
-\frac{AB}{R_1}\left[T_{11}+\left(\frac{1}{B}\frac{\partial u}{\partial\alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial\alpha_1}v\right)T_{12}-(\varphi Q_1)\right]-\\
-\frac{AB}{R_2}\left[T_{22}+\left(\frac{1}{A}\frac{\partial v}{\partial\alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial\alpha_2}u\right)T_{21}-(\psi Q_2)\right]+\\
+AB\left\{\left(\frac{1}{A}\frac{\partial w}{\partial\alpha_1}-\frac{u}{R_1}\right)\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]+\right.\\
+\frac{h}{2}\frac{\varphi}{R_1}\left[P_{s1}\left(\frac{h}{2}\right)-P_{s1}\left(-\frac{h}{2}\right)\right]+\left(\frac{1}{B}\frac{\partial w}{\partial\alpha_2}-\frac{v}{R_2}\right)\times\\
\times\left[P_{s2}\left(\frac{h}{2}\right)+P_{s2}\left(-\frac{h}{2}\right)\right]+\frac{h}{2}\frac{\psi}{R_2}\left[P_{s2}\left(\frac{h}{2}\right)-\right.\\
\left.\left.-P_{s2}\left(-\frac{h}{2}\right)\right]+\left[-P_n\left(\frac{h}{2}\right)+P_n\left(-\frac{h}{2}\right)\right]\right\}+\\
+\Delta=\rho hAB\frac{\partial^2 w}{\partial t^2}
\end{gathered}
\qquad \begin{matrix}(6.7)\\ \text{cont.}\end{matrix}
$$

Here value $\Delta$ is caused by the second component of the right side of the second formula (6.5) and is equal to

$$
\Delta=BQ_1\frac{\partial\varepsilon_{22}^0}{\partial\alpha_1}+AQ_2\frac{\partial\varepsilon_{11}^0}{\partial\alpha_2}. \qquad (6.8)
$$

From equations (6.7) it is clear that in the first two equations we can disregard values $\varphi\Delta$ and $\psi\Delta$, which are of the higher order of smallness, in the last equation from (6.7) the value $\Delta$ is of the same order as, for instance, the component $B\left(\frac{1}{A}\frac{\partial w}{\partial\alpha_1}-\frac{u}{R_1}\right)\frac{\partial T_{11}}{\partial\alpha_1}$.

Now we write the equation of moments:

$$
\begin{gathered}
\frac{\partial}{\partial\alpha_1}(BM_{11})+\frac{\partial}{\partial\alpha_1}\left[B\left(\frac{1}{B}\frac{\partial u}{\partial\alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial\alpha_1}v\right)M_{12}\right]+\\
+\frac{\partial}{\partial\alpha_2}(AM_{21})+\frac{\partial}{\partial\alpha_2}\left[A\left(\frac{1}{B}\frac{\partial u}{\partial\alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial\alpha_1}v\right)M_{22}\right]+\\
+\frac{\partial A}{\partial\alpha_2}\left[\left(\frac{1}{A}\frac{\partial v}{\partial\alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial\alpha_2}u\right)M_{11}+M_{12}\right]-\\
-\frac{\partial B}{\partial\alpha_1}\left[\left(\frac{1}{A}\frac{\partial v}{\partial\alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial\alpha_2}u\right)M_{21}+M_{22}\right]+
\end{gathered}
\qquad (6.9)
$$

$$+AB\left\{\frac{1}{R_1}\left[\left(\frac{1}{A}\frac{\partial w}{\partial \alpha_1}-\frac{u}{R_1}\right)M_{11}+\left(\frac{1}{B}\frac{\partial w}{\partial \alpha_2}-\frac{v}{R_2}\right)M_{12}\right]-\right.$$
$$-Q_1-\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)Q_2\Big\}+\frac{h}{2}AB\left\{\left[P_{s1}\left(\frac{h}{2}\right)-\right.\right.$$
$$-P_{s1}\left(-\frac{h}{2}\right)\Big]+\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)\Big[P_{s2}\left(\frac{h}{2}\right)-$$
$$-P_{s2}\left(-\frac{h}{2}\right)\Big]-\varphi\left[-P_n\left(\frac{h}{2}\right)-P_n\left(-\frac{h}{2}\right)\right]\Big\}=$$
$$=-\frac{\rho h^3AB}{12}\frac{\partial^2\varphi}{\partial t^2},$$

$$\frac{\partial}{\partial \alpha_2}(AM_{22})+\frac{\partial}{\partial \alpha_2}\left[A\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)M_{21}\right]+$$
$$+\frac{\partial}{\partial \alpha_1}(BM_{12})+\frac{\partial}{\partial \alpha_1}\left[B\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)M_{11}\right]+$$
$$+\frac{\partial B}{\partial \alpha_1}\left[\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)M_{22}+M_{21}\right]- \qquad (6.9)\ \text{cont.}$$
$$-\frac{\partial A}{\partial \alpha_2}\left[\left(\frac{1}{B}\frac{\partial u}{\partial \alpha_2}-\frac{1}{AB}\frac{\partial B}{\partial \alpha_1}v\right)M_{12}+M_{11}\right]+$$
$$+AB\left\{\frac{1}{R_2}\left[\left(\frac{1}{B}\frac{\partial w}{\partial \alpha_2}-\frac{v}{R_2}\right)M_{22}+\left(\frac{1}{A}\frac{\partial w}{\partial \alpha_1}-\frac{u}{R_1}\right)M_{21}-\right.\right.$$
$$-Q_2-\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)Q_1\Big\}+\frac{h}{2}AB\left\{\left[P_{s2}\left(\frac{h}{2}\right)-\right.\right.$$
$$-P_{s2}\left(-\frac{h}{2}\right)\Big]+\left(\frac{1}{A}\frac{\partial v}{\partial \alpha_1}-\frac{1}{AB}\frac{\partial A}{\partial \alpha_2}u\right)\Big[P_{s1}\left(\frac{h}{2}\right)-$$
$$-P_{s1}\left(-\frac{h}{2}\right)\Big]-\psi\left[-P_n\left(\frac{h}{2}\right)-P_n\left(-\frac{h}{2}\right)\right]\Big\}=$$
$$=-\frac{\rho h^3AB}{12}\frac{\partial^2\psi}{\partial t^2}.$$

In equations of moments (6.9) nonlinear terms, containing displacements of the middle surface, can be disregarded and for sloping shells we may disregard also nonlinear terms containing angles of rotation of normals $\varphi$ and $\psi$ and angles of inclination of tangents $(1/A)(\partial w/\partial \alpha)$, $(1/B)(\partial w/\partial \alpha_2)$.

In the axisymmetric deformation of the shell of rotation equations (6.7) and (6.9) assume the form:

$$\frac{\partial}{\partial s}(r_0T_{11})-\frac{\partial}{\partial s}(r_0\varphi Q)+T_{22}\sin\alpha+\frac{r_0}{R_1}\left[\left(\frac{\partial w}{\partial s}-\frac{u}{R_1}\right)T_{11}+Q\right]+$$
$$+r_0\left\{\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]-\varphi\left[-P_n\left(\frac{h}{2}\right)+P_n\left(-\frac{h}{2}\right)\right]\right\}= \qquad (6.10)$$
$$=r_0\rho h\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial}{\partial s}(r_0 Q)+\frac{\partial}{\partial s}\left[r_0\left(\frac{\partial w}{\partial s}-\frac{u}{R_1}\right)T_{11}\right]-r_0\left\{\frac{1}{R_1}[T_{11}-(\varphi Q)]+\frac{T_{22}}{R_2}\right\}+$$
$$+r_0\left\{\left(\frac{\partial w}{\partial s}-\frac{u}{R_1}\right)\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]+\frac{h\gamma}{2R_1}\left[P_{s1}\left(\frac{h}{2}\right)-\right.\right.$$
$$\left.\left.-P_{s1}\left(-\frac{h}{2}\right)\right]+\left[P_n\left(-\frac{h}{2}\right)-P_n\left(\frac{h}{2}\right)\right]\right\}=r_0\rho h\frac{\partial^2 w}{\partial t^2}, \quad (6.10) \text{ cont.}$$
$$\frac{\partial}{\partial s}(r_0 M_{11})+M_{22}\sin\alpha+r_0\left[\frac{1}{R_1}\left(\frac{\partial w}{\partial s}-\frac{u}{R_1}\right)M_{11}-Q\right]+$$
$$+\frac{r_0 h}{2}\left\{\left[P_{s1}\left(\frac{h}{2}\right)-P_{s1}\left(-\frac{h}{2}\right)\right]-\varphi\left[-P_n\left(\frac{h}{2}\right)-P_n\left(-\frac{h}{2}\right)\right]\right\}=$$
$$=-\frac{r_0\rho h^3}{12}\frac{\partial^2\gamma}{\partial t^2}.$$

Disregarding values of the order of h/L and higher as compared to 1, the motion equation can be reccrded in the form:

$$\frac{\partial T_{11}}{\partial s}-\frac{T_{11}-T_{22}}{r_0}\sin\alpha-\frac{\partial}{\partial s}(\varphi Q)+\frac{1}{r_0}(\varphi Q)\sin\alpha+\frac{Q}{R_1}+$$
$$+\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]-\varphi\left[-P_n\left(\frac{h}{2}\right)+P_n\left(-\frac{h}{2}\right)\right]=$$
$$=\rho h\frac{\partial^2 u}{\partial t^2}, \quad (6.11)$$
$$\frac{\partial Q}{\partial s}-\frac{1}{r_0}Q\sin\alpha+\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial s}T_{11}\right)-\frac{1}{r_0}\left(\frac{\partial w}{\partial s}T_{11}\right)\sin\alpha-$$
$$-\left[\frac{1}{R_1}(T_{11}-\varphi Q)+\frac{T_{22}}{R_2}\right]+\frac{\partial w}{\partial s}\left[P_{s1}\left(\frac{h}{2}\right)+P_{s1}\left(-\frac{h}{2}\right)\right]+$$
$$+\left[-P_n\left(\frac{h}{2}\right)+P_n\left(-\frac{h}{2}\right)\right]=\rho h\frac{\partial^2 w}{\partial t^2},$$
$$\frac{\partial M_{11}}{\partial s}-\frac{M_{11}-M_{22}}{r_0}\sin\alpha-Q+$$
$$+\frac{h}{2}\left[P_{s1}\left(\frac{h}{2}\right)-P_{s1}\left(-\frac{h}{2}\right)\right]=-\frac{\rho h^3}{12}\frac{\partial^2\gamma}{\partial t^2}.$$

## § 7. Equations of Motion In Displacements In Axisymmetric Deformation

We introduce abbreviated designations:

$$E_{11}=\varepsilon_{11}^0+\frac{1}{2}\varepsilon_{22}^0,\ E_{22}=\varepsilon_{22}^0+\frac{1}{2}\varepsilon_{11}^0,\ K_{11}=k_{11}+$$
$$+\frac{1}{2}k_{22},\ K_{22}=k_{22}+\frac{1}{2}k_{11}; \quad (7.1)$$
$$F_{uw}=\frac{\partial}{\partial s}(x_1, w)-w_s(x_1, u), F_{uk}=-\frac{\partial}{\partial s}(x_1, u)-k_{11}; \quad (7.2)$$

$$\tilde{D}_\varepsilon = \frac{\partial \varepsilon_{22}^0}{\partial s} = -\frac{1}{r_0}[\sin\alpha(\varepsilon_{11}^0 - \varepsilon_{22}^0) - \cos\alpha(w_s - \varkappa_1 u)]; \tag{7.3}$$

$$\tilde{D}_k = \frac{\partial k_{22}}{\partial s} = -\frac{1}{r_0}[\sin\alpha(k_{11} - k_{22}) + \varkappa_1 \varphi\cos\alpha]; \tag{7.4}$$

$$E_{1z} = w_s - \varkappa_1 u. \tag{7.5}$$

In this case derivatives from quadratic forms $P_0$, $P_1$, $P_2$ on coordinate s along the meridian will be equal to:

$$\begin{aligned}
\frac{\partial P_0}{\partial s} &= 2E_{11}u_{ss} + \left(2E_{11}E_{1z} + \frac{1}{2}\varepsilon_{1z}\right)w_{ss} + \\
&+ 2E_{11}F_{uw} + 2E_{22}\tilde{D}_\varepsilon + \frac{1}{2}\varepsilon_{1z}F_{uk}. \\
\frac{\partial P_1}{\partial s} &= K_{11}u_{ss} + K_{11}E_{1z}w_{ss} + E_{11}\varphi_{ss} + \\
&+ K_{11}F_{uw} + K_{22}\tilde{D}_\varepsilon + E_{22}\tilde{D}_k. \\
\frac{\partial P_2}{\partial s} &= 2K_{11}\varphi_{ss} + 2K_{22}\tilde{D}_k.
\end{aligned} \tag{7.6}$$

Therefore, derivatives from integrals $I_k$ ($k = 1, 2, 3$) on coordinate s can be determined by the formula

$$\begin{aligned}
\frac{\partial I_k}{\partial s} &= (2E_{11}I_{ko} + K_{11}I_{k1})u_{ss} + \left[\left(2E_{11}E_{1z} + \frac{1}{2}\varepsilon_{1z}\right)I_{ko} + \right. \\
&\left. + K_{11}E_{1z}I_{k1}\right]w_{ss} + (E_{11}I_{k1} + 2K_{11}I_{k2})\varphi_{ss} + 2I_{ko}\Big(E_{11}E_{uw} + \\
&+ E_{22}\tilde{D}_\varepsilon + \frac{1}{4}\varepsilon_{1z}F_{uk}\Big) + I_{k1}(K_{11}F_{uw} + K_{22}\tilde{D}_\varepsilon + E_{22}\tilde{D}_k) + \\
&+ 2I_{k2}K_{22}\tilde{D}_k.
\end{aligned} \tag{7.7}$$

After substitution of expressions of moments, forces, intersecting forces and their derivatives with respect to coordinate in motion equations (6.11) and simple transformations we obtain a system of three quasi-linear second-order equations with respect to φ, w and u:

$$\begin{aligned}
\varphi_{tt} &= a_1\varphi_{ss} + a_2 w_{ss} + a_3 u_{ss} + L, \\
w_{tt} &= b_1\varphi_{ss} + b_2 w_{ss} + b_3 u_{ss} + M, \\
u_{tt} &= c_1\varphi_{ss} + c_2 w_{ss} + c_3 u_{ss} + N.
\end{aligned} \tag{7.8}$$

Let us note that equation of moments is linear with respect to

coefficients $a_1$, ..., $b_1$, ..., $c_1$, ..., L, M, N, and the force equation with respect to these coefficients is nonlinear. Bearing in mind that in motion equations in projection on the direction of meridian with small intersecting forces we can disregard nonlinear terms, it is expedient to present coefficients $b_1$, $b_2$, $b_3$, M and $c_1$, $c_2$, $c_3$, N as consisting of two parts:

$$\begin{gathered} b_1 = b_1^0 + b_1', \quad b_2 = b_2^0 + b_2', \quad b_3 = b_3^0 + b_3', \quad M = M^0 + M', \\ c_1 = c_1^0 + c_1', \quad c_2 = c_2^0 + c_2', \quad c_3 = c_3^0 + c_3', \quad N = N^0 + N', \end{gathered} \tag{7.9}$$

where strokes designate components, dependent on nonlinear terms in force equations.

We introduce the following designations for the sake of brevity in writing:

$$\begin{gathered} \Delta_s = 2\left(E_{11}F_{uw} + E_{22}\tilde{D}_t + \frac{1}{4}\varepsilon_{12}F_{uk}\right)(E_{11}I_{20} - K_{11}I_{30}) + \\ + (K_{11}F_{uw} + K_{22}\tilde{D}_t + E_{22}\tilde{D}_k)(E_{11}I_{21} - K_{11}I_{31}) + 2K_{22}\tilde{D}_k(E_{11}I_{22} - \\ - K_{11}I_{32}) + I_2\left(F_{uw} + \frac{1}{2}\tilde{D}_t\right) - \frac{1}{2}I_3\tilde{D}_k. \\ \Delta_Q = \varepsilon_{12}\left[2\left(E_{11}F_{uw} + E_{22}\tilde{D}_t + \frac{1}{4}\varepsilon_{12}F_{uk}\right)I_{10} + \right. \\ \left. + (K_{11}F_{uw} + K_{22}\tilde{D}_t + E_{22}\tilde{D}_k)I_{11} + 2K_{22}\tilde{D}_kI_{12}\right] + I_1F_{uk}, \\ \Delta_T = 2\left(E_{11}F_{uw} + E_{22}\tilde{D}_t + \frac{1}{4}\varepsilon_{12}F_{uk}\right)(E_{11}I_{10} - K_{11}I_{20}) + \\ + (K_{11}F_{uw} + K_{22}\tilde{D}_t + E_{22}\tilde{D}_k)(E_{11}I_{11} - K_{11}I_{21}) + \\ + 2K_{22}\tilde{D}_k(E_{11}I_{12} - K_{11}I_{22}) + I_1\left(F_{uw} + \frac{1}{2}\tilde{D}_t\right) - \frac{1}{2}I_2\tilde{D}_k. \end{gathered} \tag{7.10}$$

Then for coefficients of the system of equations (7.8) we obtain the following formulas:

$$\begin{gathered} a_1 = \frac{4}{3}[I_3 - E_{11}(E_{11}I_{21} - K_{11}I_{31}) - 2K_{11}(E_{11}I_{22} - K_{11}I_{32})]\frac{1}{\rho h} \\ a_2 = -\frac{4}{3}\left[I_2E_{12} + 2\left(E_{11}E_{12} + \frac{1}{4}\varepsilon_{12}\right)(E_{11}I_{20} - K_{11}I_{30}) + \right. \\ \left. + K_{11}E_{12}(E_{11}I_{21} - K_{11}I_{31})\right]\frac{1}{\rho h}, \\ a_3 = -\frac{4}{3}\left[I_2 + 2E_{11}(E_{11}I_{20} - K_{11}I_{30}) + K_{11}(E_{11}I_{21} - K_{11}I_{31})\right]\frac{1}{\rho h}, \end{gathered} \tag{7.11}$$

$$L = \frac{4}{3}\left\{\frac{1}{r_0}(M_{11} - M_{22})\sin\alpha + Q - \Delta_M - \frac{h}{2}\left[P_s\left(\frac{h}{2}\right) - P_s\left(-\frac{h}{2}\right)\right]\right\}\frac{1}{\rho h}; \tag{7.11 cont.}$$

$$b_1^0 = \frac{1}{3}\varepsilon_{12}(E_{11}I_{11} + 2K_{11}I_{12})\frac{1}{\rho h},$$

$$b_2^0 = \frac{1}{3}\left[I_1 + 2\varepsilon_{12}\left(E_{11}E_{12} + \frac{1}{4}\varepsilon_{12}\right)I_{10} + K_{11}E_{12}I_{11}\right]\frac{1}{\rho h},$$

$$b_3^0 = \frac{1}{3}\varepsilon_{12}(2E_{11}I_{10} + K_{11}I_{11}). \tag{7.12}$$

$$M_0 = \frac{1}{3}\left\{-3\left(\frac{1}{r_0}Q\sin\alpha + \varkappa_1 T_{11} + \varkappa_2 T_{22}\right) + \left[-P_n\left(\frac{h}{2}\right) + P_n\left(-\frac{h}{2}\right)\right] + \Delta_Q\right\};$$

$$c_1^0 = \frac{4}{3}[-I_2 + E_{11}(E_{11}I_{11} - K_{11}I_{21}) + 2K_{11}(E_{11}I_{12} - K_{11}I_{22})]\frac{1}{\rho h},$$

$$c_2^0 = \frac{4}{3}\left[I_1E_{12} + 2\left(E_{11}E_{12} + \frac{1}{4}\varepsilon_{12}\right)(E_{11}I_0 - K_{11}I_{20}) + K_{11}E_{12}(E_{11}I_{21} - K_{11}I_{31})\right]\frac{1}{\rho h},$$

$$c_3^0 = \frac{4}{3}[I_1 + 2E_{11}(E_{11}I_{10} - K_{11}I_{20}) + K_{11}(E_{11}I_{11} - K_{11}I_{21})]\frac{1}{\rho h},$$

$$N^0 = \left\{-\left[\frac{1}{r_0}(T_{11} - T_{22})\sin\alpha - \varkappa_1 Q\right] + \left[P_s\left(\frac{h}{2}\right) + P_s\left(-\frac{h}{2}\right)\right] + \frac{4}{3}\Delta_T\right\}\frac{1}{\rho h}; \tag{7.13}$$

$$b_1' = w_s c_1^0, \quad b_2' = w_s c_2^0 + \frac{T_{11}}{\rho h},$$

$$b_3' = w_s c_3^0, \quad M' = \frac{4}{3}w_s\frac{\Delta_T}{\rho h},$$

$$c_1' = -\varphi b_1^0, \quad c_2' = -\varphi b_2^0,$$

$$c_3' = -\varphi b_3^0; \tag{7.14}$$

$$N' = \left\{-K_{11}Q + \frac{1}{r_0}(\varphi Q)\sin\alpha - \varphi\left[-P_n\left(\frac{h}{2}\right) + P_n\left(-\frac{h}{2}\right)\right] - \frac{1}{3}\varphi\Delta_Q\right\}\frac{1}{\rho h}. \tag{7.15}$$

For elastic-deformations integrals $I_k$ ($k = 1, 2, 3$) are constants: $I_1 = Eh$, $I_2 = 0$, $I_3 = Eh^3/2$, and therefore, their derivatives $I_{kj} = 0$ ($j = 0, 1, 2$).

Thus, the first motion equation will be linear, the other two — quasi-linear, where

$$\Delta_M^{(e)} = -\frac{\nu}{12}Eh^3\tilde{D}_k, \quad \Delta_Q^{(e)} = IhF_{uk}, \quad \Delta_T^{(e)} = Eh(F_{uw} + \nu\tilde{D}_s). \tag{7.16}$$

Coefficients of motion equations for elastic deformations will be equal:

$$a_1 = \frac{1}{1-\nu^2}\frac{E}{\rho},\ a_2 = a_3 = 0,\ L = \frac{1}{1-\nu^2}\left[\frac{1}{r_0}(M_{11} - M_{22})\sin\alpha + + Q - \frac{\nu}{12}Eh^2\tilde{D}_k\right]; \tag{7.17}$$

$$b_1^0 = 0,\ b_2^0 = \frac{l}{\rho},\ b_3^0 = 0,$$

$$M^0 = -\left[\frac{1}{r_0}Q\sin\alpha + \varkappa_1 T_{11} + \varkappa_2 T_{22}\right] + + \left[-P_n\left(\frac{h}{2}\right) + P_n\left(-\frac{h}{2}\right)\right] + lhF_{uk}; \tag{7.18}$$

$$c_1^0 = c_2^0 = 0,\ c_3^0 = \frac{1}{1-\nu^2}\frac{E}{\rho},$$

$$N^0 = \left[\frac{1}{r_0}(T_{11} - T_{22})\sin\alpha - \varkappa_1 Q\right] + \left[P_s\left(\frac{h}{2}\right) + P_s\left(-\frac{h}{2}\right)\right] + + \frac{Eh}{1-\nu^2}(F_{uw} + \nu\tilde{D}_s); \tag{7.19}$$

$$c_1' = 0,\ c_2' = -\varphi\frac{l}{\rho},\ c_3' = 0,$$

$$b_1' = 0,\ b_2' = \frac{T_{11}}{\rho h}, \tag{7.20}$$

$$b_3 = \frac{E}{1-\nu^2}w_s,\ M' = \frac{E}{1-\nu^2}(F_{uw} + \nu\tilde{D}_s)\,w_s;$$

$$N' = \left\{-K_{11}Q + \frac{1}{r_0}(\varphi Q)\sin\alpha - \varphi\left[-P_n\left(\frac{h}{2}\right) + P_n\left(-\frac{h}{2}\right)\right] - - \frac{1}{3}\varphi lhF_{uk}\right\}\frac{1}{\rho h}. \tag{7.21}$$

For the unloading throughout the entire thickness of the shell made from incompressible material motion equations will have this form:

$$\begin{aligned} \varphi_{tt} &= a_1\varphi_{ss} + L^*, \\ w_{tt} &= b_2 w_{ss} + b_3 u_{ss} + M^*, \\ u_{tt} &= c_2 w_{ss} + c_3 u_{ss} + N^*, \end{aligned} \tag{7.22}$$

where

$$a_1 = \frac{4}{3}\frac{E}{\rho},\ L^* = \tilde{\varphi}_{tt} - a_1\tilde{\varphi}_{ss} + L_e - \tilde{L}_e;$$

$$b_2 = \frac{l}{\rho} + \frac{T_{11}}{\rho h},\ b_3 = \frac{4}{3}w_s,$$

$$M^* = \tilde{w}_{tt} - b_2\tilde{w}_{ss} - b_3\tilde{u}_{ss} + M_e - \tilde{M}_e; \tag{7.23}$$

$$c_1 = -\varphi\frac{l}{\rho}, \quad c_3 = \frac{4}{3}\frac{E}{\rho}. \tag{7.24}$$

$$N^* = \tilde{u}_{11} - c_2 w_{33} - c_3\tilde{u}_{33} + N_e - \tilde{N}_e. \tag{7.25}$$

Here $L_e$, $M_e$, $N_e$ are coefficient values for elastic deformations when $t \leq \tilde{t} + 0$, and sign ($\sim$) marks values of corresponding functions in the moment of the beginning of unloading $t = \tilde{t} - 0$.

## § 8. Initial and Boundary Conditions

If on the surface, which limits the body, external forces f* are given, projections of which on coordinate lines $\alpha_1$, $\alpha_2$, $\alpha_3$ are equal $f_1^*$, $f_2^*$, $f_3^*$, then the following conditions should be fulfilled on the surface:

$$\begin{aligned} S_{11}\cos(nk_1) + S_{21}\cos(nk_2) + S_{31}\cos(nk_3) &= f_1^*, \\ S_{12}\cos(nk_1) + S_{22}\cos(nk_2) + S_{32}\cos(nk_3) &= f_2^*, \\ S_{13}\cos(nk_1) + S_{23}\cos(nk_2) + S_{33}\cos(nk_3) &= f_3^*. \end{aligned} \tag{8.1}$$

Here $(nk_1)$, $(nk_2)$, $(nk_3)$ are angles, formed by the normal to the undeformed surface with directions $\alpha_1$, $\alpha_2$, $\alpha_3$.

If however external forces "watch" the deformed surface of a body (for instance, pressure of liquid or gas) and projections of the external force f* on directions 1', 2', 3' of axes 1, 2, 3 after deformations are equal to $f_1'$, $f_2'$, $f_3'$, then condition on the surface will have this form:

$$\begin{aligned} &[(1+2\varepsilon_{11})\sigma^*_{11} + \varepsilon_{12}\sigma^*_{12} + \varepsilon_{13}\sigma^*_{13}]\cos(nk_1) + [(1+2\varepsilon_{11})\sigma^*_{12} + \varepsilon_{12}\sigma^*_{22} + \\ &+ \varepsilon_{13}\sigma^*_{23}]\cos(nk_2) + [(1+2\varepsilon_{11})\sigma^*_{13} + \varepsilon_{12}\sigma^*_{23} + \varepsilon_{13}\sigma^*_{33}]\cos(nk_3) = \\ &= \frac{S_n^*}{S_n}\left[(1+e_{11})f_1^* + \left(\frac{1}{2}e_{12} + \omega_3\right)f_2^* + \left(\frac{1}{2}e_{13} - \omega_2\right)f_3^* = f_1', \right. \\ &[(1+2\varepsilon_{22})\sigma^*_{22} + \varepsilon_{23}\sigma^*_{23} + \varepsilon_{21}\sigma^*_{21}]\cos(nk_2) + [(1+2\varepsilon_{22})\sigma^*_{23} + \varepsilon_{23}\sigma^*_{33} + \\ &+ \varepsilon_{21}\sigma^*_{31}]\cos(nk_3) + [(1+2\varepsilon_{22})\sigma^*_{21} + \varepsilon_{21}\sigma^*_{11} + \varepsilon_{23}\sigma^*_{31}]\cos(nk_1) = \\ &= \frac{S_n^*}{S_n}\left[(1+e_{22})f_2^* + \left(\frac{1}{2}e_{23} + \omega_1\right)f_3^* + \left(\frac{1}{2}e_{21} - \omega_3\right)f_1^*\right] = f_2', \end{aligned} \tag{8.2}$$

$$[(1+2\varepsilon_{33})\sigma^*_{z3}+\varepsilon_{z1}\sigma^*_{z1}+\varepsilon_{z2}\sigma^*_{z2}]\cos(nk_3)+[(1+2\varepsilon_{33})\sigma^*_{z1}+\varepsilon_{z1}\sigma^*_{11}+$$
$$+\varepsilon_{z2}\sigma^*_{12}]\cos(nk_1)+[(1+2\varepsilon_{33})\sigma^*_{z2}+\varepsilon_{z1}\sigma^*_{12}+\varepsilon_{12}\sigma^*_{22}]\cos(nk_2)= \quad (8.2 \text{ cont'd})$$
$$=\frac{S_n^*}{S_n}\left[(1+e_{33})f_3^*+\left(\frac{1}{2}e_{z1}+\omega_2\right)f_1^*+\left(\frac{1}{2}e_{z2}-\omega_1\right)f_2^*\right]=f_3^*.$$

For small deformations relationships (8.2) will be simplified and will assume this form:

$$\begin{aligned}
\sigma_{11}\cos(nk_1)+\sigma_{12}\cos(nk_2)+\sigma_{1z}\cos(nk_3)&=f_1^*,\\
\sigma_{21}\cos(nk_1)+\sigma_{22}\cos(nk_2)+\sigma_{2z}\cos(nk_3)&=f_2^*,\\
\sigma_{z1}\cos(nk_1)+\sigma_{z2}\cos(nk_2)+\sigma_{zz}\cos(nk_3)&=f_3^*.
\end{aligned} \quad (8.3)$$

On the surface of a body displacements can also be given

$$u = u(t), \quad v = v(t), \quad w = w(t).$$

In solving dynamic problems in points of boundary surface we may be given, speeds and accelerations, and not displacements.

Let us first consider the unclosed shell, the contour of the middle surface of which is described by equation $F(\alpha_1, \alpha_2) = 0$. Let us assume that to this contour force $\overline{K}^*$ and moment $\overline{G}^*$ are applied.

Let us expand vector $\overline{K}^*$ in the directions $\alpha_1$, $\alpha_2$, z:

$$\overline{K}^* = K_1^*\overline{k}_1 + \overline{K}_2^*\overline{k}_2 + K_3^*\overline{k}_3. \quad (8.4)$$

The connection between internal forces $T_{ik}$ and $Q_i$ and components $K_1^*$, $K_2^*$, $K_3^*$ of $\overline{K}^*$ can be obtained by integrating (8.1) with respect to thickness, in this we should take into consideration that

$$K_1^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_1^* dz, \quad K_2^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_2^* dz, \quad K_3^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_3^* dz;$$

as a result, for a small deformation, we obtain ($\alpha_1 = \alpha$, $\alpha_2 = \beta$):

$$\left[T_{11}+\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)T_{12}-\varphi Q_1\right]\cos(nk_1)+$$
$$+\left[T_{21}+\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)T_{22}-\varphi Q_2\right]\cos(nk_2)=K_1^*. \quad (8.5)$$

$$\left[T_{22}+\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)T_{21}-\psi Q_2\right]\cos(nk_2)+$$
$$+\left[T_{12}+\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)T_{11}-\psi Q_1\right]\cos(nk_1)=K_2^{*},$$
$$\left[Q_1+\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)T_{11}+\left(\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2}\right)T_{12}\right]\cos(nk_1)+$$
$$+\left[Q_2+\left(\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2}\right)T_{22}+\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)T_{21}\right]\cos(nk_2)=K_3^{*}. \qquad (8.5\ \text{cont'd})$$

If the shell is limited by lines $\alpha = c_i$ = const and $\beta = c_j$ = = const ($i, j$ = 1, 2), to which forces $K_{1i}^{*}$, $K_{2i}^{*}$, $K_{3i}^{*}$, $K_{1j}^{*}$, $K_{2j}^{*}$, $K_{3j}^{*}$, are applied then on lines $\alpha = c_i$ = const these conditions must be fulfilled

$$T_{11}+\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)T_{12}-\varphi Q_1=K_{1i}^{*},$$
$$T_{12}+\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)T_{11}-\psi Q_1=K_{2i}^{*}, \qquad (8.6)$$
$$Q_1+\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)T_{11}+\left(\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2}\right)T_{12}=K_{3i}^{*},$$

and on lines $\beta = c_j$ = const — conditions:

$$T_{21}+\left(\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)T_{12}-\varphi Q_2=K_{1j}^{*},$$
$$T_{22}+\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)T_{21}-\psi Q_2=K_{2j}^{*},$$
$$Q_2+\left(\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{v}{R_2}\right)T_{22}+\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)T_{21}=K_{3j}^{*}. \qquad (8.6')$$

Here $K_{1i}^{*}$, $K_{2j}^{*}$, are normal forces, $K_{2i}^{*}$, $K_{2j}^{*}$, are tangent forces, $K_{3i}^{*}$, $K_{3j}^{*}$, are intersecting forces, acting on the contour. For sloping shells nonlinear terms, containing displacements of the middle surface u and v, can be disregarded.

If surface forces, in the process of deformation "watch" the direction of the normal and tangent to the contour and are equal respectively, to $K_1'$, $K_2'$, $K_3'$ in small deformations from (8.3) we obtain boundary conditions for the forces in that same form as in small sags, namely:

$$T_{11}\cos(nk_1) + T_{12}\cos(nk_2) = K_1^*,$$
$$T_{22}\cos(nk_2) + T_{21}\cos(nk_1) = K_2^*,$$
$$Q_1\cos(nk_1) + Q_2\cos(nk_2) = K_3^*. \quad (8.7)$$

For moments in small deformations we do not have to distinguish between projections of the vector of moments on the deformed and undeformed contour of the shell, and therefore, boundary conditions for moments will be written in this manner:

$$M_{11}\cos(nk_1) + M_{12}\cos(nk_2) = G_1^*,$$
$$M_{22}\cos(nk_2) + M_{21}\cos(nk_1) = G_2^*. \quad (8.8)$$

If the shell is limited by coordinate lines $\alpha = c_i = \text{const}$ and $\beta = c_j = \text{const}$, then along $\alpha$ –

$$M_{11} = G_{1i}^*, \quad M_{21} = G_{2i}^*, \quad (8.9)$$

and along lines $\beta$ –

$$M_{12} = G_{1j}^*, \quad M_{22} = G_{2j}^*. \quad (8.10)$$

Here $G_{1i}^*$, $G_{2j}^*$ are bending moments; $G_{2i}^*$, $G_{1j}^*$ – torques, acting on the contour of the shell.

Along the entire contour of the shell or its part, instead of forces and moments we can be given displacements of the middle surface u, v, sag w and angles of rotation of normals $\varphi$ and $\psi$ or their first or second derivatives.

Thus, on every section of the shell contour five boundary conditions have to be given, which is in full conformity with the available five motion equations with respect to five unknown functions u, v, w, $\varphi$, $\psi$.

If the shell presents a ruled surface with a closed directrix, where $\alpha$ is the coordinate along generators, and $\beta$ is the coordinate along the directrix, then when $\alpha = c_i$ $(i = 1, 2)$ ten conditions of the single-value of displacements and their derivatives with respect to

β must be fulfilled, and for the shell of rotation, when coordinate ω is the azimuth, they will have the form:

$$\begin{gathered} u(\omega)=u(\omega+2\pi),\ v(\omega)=v(\omega+2\pi),\ w(\omega)=w(\omega+2\pi), \\ \varphi(\omega)=\varphi(\omega+2\pi),\ \psi(\omega)=\psi(\omega+2\pi), \\ u_2'(\omega)=u_2'(\omega+2\pi),\ v_2'(\omega)=v_2'(\omega+2\pi), \\ w_2'(\omega)=w_2'(\omega+2\pi), \\ \varphi_2'(\omega)=\varphi_2'(\omega+2\pi),\ \psi_2'(\omega)=\psi_2'(\omega+2\pi). \end{gathered} \tag{8.11}$$

For dynamic problems it is sometimes expedient to use the condition of single value of derivatives from time and β coordinate displacements.

The remaining ten equations will supply immobilizing conditions when $\alpha = c_1$ and $\alpha = c_2$.

If the deformation of the shell of rotation is symmetrical with respect to the plane, passing through points $\omega = 0$ and $\omega = \pi$, then it will be possible to consider only the section $0 \leq \omega \leq \pi$, and instead of conditions of periodicity to use ten conditions of symmetry, when $\omega = 0$ and $\omega = \pi$

$$v=0,\ \psi=0,\ \frac{\partial w}{\partial \beta}=0,\ \frac{\partial u}{\partial \beta}=0,\ \frac{\partial \varphi}{\partial \beta}=0, \tag{8.12}$$

where instead of the third condition, taking into account the second condition it is possible to take condition of symmetry $Q_2 = 0$. For a shell, closed on both coordinates α and β, conditions of single value both for α and β must be fulfilled.

## CHAPTER II

## ELASTIC OSCILLATIONS OF SHELLS

### §1. Natural Oscillations. Formulation of Problem

Let us set up a problem on thin-shell oscillations according to Love [6]. It is known that equations for shell equilibrium are obtained by means of equating to zero the main vector and the moment of all forces, applied to any part thereof. Equations for shell oscillations can be set up by means of addition of expressions for forces of inertia and their moments to external forces and pairs, which enter in equations of equilibrium:*

$$-2\rho h\frac{\partial^2 u}{\partial t^2},\quad -2\rho h\frac{\partial^2 v}{\partial t^2},\quad -2\rho h\frac{\partial^2 w}{\partial t^2},$$

where $\rho$ is density of material.

In setting up the equations we reject all products of values u, v, w by their derivatives; since forces and moments are linear functions of these values, we will simplify the equations, referring them to the undeformed state of the shell. Equations of moments we will write in the following form:

*Damping forces are not examined here.

$$\frac{1}{AB}\left\{\frac{\partial(M_{12}B)}{\partial\alpha}-\frac{\partial(M_2A)}{\partial\beta}+M_1\frac{\partial A}{\partial\beta}-M_{21}\frac{\partial B}{\partial\alpha}\right\}+N_2\quad 0.$$

$$\frac{1}{AB}\left\{\frac{\partial(M_1B)}{\partial\alpha}+\frac{\partial(M_{21}A)}{\partial\beta}-M_{12}\frac{\partial A}{\partial\beta}-M_2\frac{\partial B}{\partial\alpha}\right\}-N_1=0.$$

$$\frac{M_{12}}{R_1}+\frac{M_{21}}{R_2}+S_1+S_2=0 \qquad (1.1)$$

and the three equations for forces will be:

$$\frac{1}{AB}\left\{\frac{\partial(T_1B)}{\partial\alpha}-\frac{\partial(S_2A)}{\partial\beta}+S_1\frac{\partial A}{\partial\beta}-T_2\frac{\partial B}{\partial\alpha}\right\}-\frac{N_1}{R_1}=2\rho h\frac{\partial^2u}{\partial t^2},$$

$$\frac{1}{AB}\left\{\frac{\partial(S_1B)}{\partial\alpha}+\frac{\partial(T_2A)}{\partial\beta}-T_1\frac{\partial A}{\partial\beta}-S_2\frac{\partial B}{\partial\alpha}\right\}-\frac{N_2}{R_2}=2\rho h\frac{\partial^2v}{\partial t^2},$$

$$\frac{1}{AB}\left\{\frac{\partial(N_1B)}{\partial\alpha}+\frac{\partial(N_2A)}{\partial\beta}\right\}+\frac{T_1}{R_1}+\frac{T_2}{R_2}=2\rho h\frac{\partial^2w}{\partial t^2}. \qquad (1.2)$$

Equations (1.2) constitute a system of oscillation equations, where some of the values included are connected by relationships (1.1).

These equations must be transformed into a system of partial differential equations for determination of displacements u, v, w by means of replacement of values $T_1$, ..., by expressions using u, v, w and their derivatives, while the third equation from (1.1) should turn into identity.

Let us note that, as a particular case, the theory of oscillation of plane plates is included here. Actually, if one were to assume that $\frac{1}{R_1}=\frac{1}{R_2}=0$ in all equations (1.1) and (1.2), then these equations will fall into two groups: one of them will contain $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 v}{\partial t^2}$ and force T, S, the other $\frac{\partial^2 w}{\partial t^2}$, the elastic force N and moments M. Further, in this case T, S are expressed through $\varepsilon_1$, ..., and the latter in turn through u, v according to known formulas.

Thus, one of the groups, into which equations (1.1) and (1.2) are divided, is identical with equations of longitudinal oscillations (deformations are reduced to elongation in plane of the plate). Further, moments M are expressed through $\varkappa_1$, ..., and the latter — through w according to known formulas. Components $N_1$, and $N_2$ are expressed through moments M by the formulas:

$$N_1 = \frac{\partial M_1}{\partial \alpha} + \frac{\partial M_{21}}{\partial \beta}, \quad N_2 = \frac{\partial M_2}{\partial \beta} - \frac{\partial M_{12}}{\partial \alpha}.$$

This second group of equations is equivalent to the equation for transverse oscillations of the plate.

With such a rendition of the theory of oscillations we make assumptions, similar to asumptions, applied in the theory of thin rods. We assume that the deformed state in the thin oscillating shell (or plate) is of the same type, as that determined in setting up equations of equilibrium. For instance, in the case of the plane plate [3], subjected to transverse oscillations, we make an assumption that internal deformation in a small part of the plate is very close to that form of deformation, which this part would have, if it were kept in equilibrium with the same degree of distortion of the middle plane. Let us consider the state of cylindrical or prismatic element of plane plate, inserted in a corresponding hole in it. Let us assume that during transverse oscillations such an element of the plate in any moment of the period of oscillations is practically in the same state, as in equilibrium. If this takes place, then deformation component in this section during transverse oscillations will be equal to:

$$e_{xx} = -z\varkappa_1, \; e_{yy} = -z\varkappa_2, \; e_{xy} = -2z\tau, \; e_{zz} = \frac{\nu}{1-\nu} z(\varkappa_1 + \varkappa_2).$$

and when the plate oscillates in its own plane,

$$e_{xx} = e_1, \; e_{yy} = e_2, \; e_{xy} = \omega, \; e_{zz} = \frac{-\nu}{1-\nu} \cdot (e_1 + e_2).$$

In both cases $e_{zz}$ is such that stress $Z_z$ is equal to zero. It is clear that our assumption is justified, when the period of oscillation of the plate is great as compared to the period of those free oscillations of the prismatic element of the plate, with which the deformation is of the assumed type. Actually, the period of all transverse oscillations of a plate is directly proportional to the square of the linear dimension of the area, included in the outline of the plate, and is inversely proportional to its thickness; the period of any kind of longitudinal oscillations is directly proportional to the linear dimensions of plate and does not depend on its thickness. The period of any free oscillations of the prismatic element, accompanied by deformations of the type adopted here, is proportional to the linear dimensions of this element or approximately proportional to the thickness of the plate. In this reasoning there is nothing which specially pertains to the plane plate only. Hence, we conclude that in an oscillating plate or shell the deformed state in the small section must be considered to be practically the same, as if the plate were in equilibrium, during which the middle surface would have such stretch and bend, as in any moment during oscillation. It should be borne in mind also that these reasonings, which justify the assumption made, become invalid when oscillation frequency increases.

Displacement component must satisfy equations (1.2), which are transformed, as it is indicated above. Furthermore, they must satisfy the boundary conditions. On the free ends the bending pair, and the three linear combinations, composed of forces and the turning pair, must turn into zero.

Let us note that expression for moments M, of forces N contain factor $D = \frac{Eh^3}{12(1 - \nu^2)}$ or $\frac{2}{3}\frac{Eh^3}{1 - \nu^2}$, and expressions for the forces consist of two members, of which one is proportional to h and the other to $h^3$. Each of equations (1.2) we shall divide by h; then terms depending on $\varepsilon_1$, $\varepsilon_2$, and w, will not contain h, while others will contain factor $h^2$. Further, we assume that we can obtain a correct approximate solution, rejecting terms, containing $h^2$. If we do this then on the free edges two boundary conditions, namely, $M_i = 0$ and $N - \frac{\partial M_{ij}}{\partial s} = 0$ become superfluous; the system of equations will be of a sufficiently high order to satisfy the remaining boundary conditions. But, now h is left out of the equations and boundary conditions and, therefore, frequency will not depend on thickness. Lengthening of the middle surface will be the most important feature of the deformation, and, furthermore, deformation is necessarily accompanied by a bend.

Oscillations of thin shells, accompanied by elongations, are analogous to oscillations of this type for plane plates. Examination of shells with slightly bent middle surfaces shows that an open shell can accomplish such kind of oscillations, which are analogous to transverse oscillations of plane plates. The frequency of these oscillations will be significantly lower than the frequency of oscillations, during which elongation of the middle surface occurs. The existence of such kind of oscillations may be established by means of the following reasonings.

The upper limit for the lowest pitch frequency can be found, if we set out to achieve a certain suitable type of oscillation, since in an oscillating system the frequency, obtained for an adopted type

of oscillations, cannot be less than the lowest frequency of natural oscillations. If, for instance, we adopt such a type of oscillation, with which lines, drawn on the middle surface, do not change their own length, we can calculate the frequency with the help of a formula for the kinetic and potential bending energies. Since the kinetic energy is proportional to h, and potential energy is proportional to $h^3$, then the frequency should be proportional to h. Frequency of similar oscillations, not accompanied by elongations in a shell of a given shape decreases indefinitely together with h in contrast to longitudinal oscillations. It follows from this that the frequency of longitudinal oscillations cannot be the lowest. However, let us note that the case of the closed shell, for instance a spherical shell, is an exception, since here oscillations without elongations are absolutely impossible: similarly a shell of small thickness, which is almost closed and has only a small hole is also included in this exception, but only if this hole is sufficiently small. In order to force the shell to oscillate in such a manner that there would be no elongations, it will be necessary to apply forces to its edges and its surface. If these forces are absent, then the displacement differs from the displacement, which satisfies conditions of deformation without elongations. However, this difference for low oscillation frequencies should be insignificant, since otherwise we would have to deal in actual practice with longitudinal oscillations and the frequency in reality could not be sufficiently small, to correspond to the given case. As we can conclude from the form of motion equations, the elongation, which we are discussing, on the greater part of the surface is extremely small; only near the edges will it be such, as to satisfy the condition on these boundaries.

## §2. The Closed Cylindrical Shell

Let us assume that a is the radius of a shell, and $\alpha = x$, $\beta = \varphi$. Let us assume that the edges of the shell are formed by two circumferences $x = \pm l$. Elongations and change of curvatures are determined by values:

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \frac{1}{a}\left(\frac{\partial v}{\partial \varphi} - w\right), \quad \omega = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \varphi},$$

$$\varkappa_1 = \frac{\partial^2 w}{\partial x^2}, \quad \varkappa_2 = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial v}{\partial \varphi}\right), \quad \tau = \frac{1}{a}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \varphi} + v\right).$$

The displacement is a periodic function with respect to $\varphi$ with period of $2\pi$. It is assumed that normal oscillations of the shell have a frequency of $\frac{p}{2\pi}$. Therefore, we assume that u, v, and w are proportional to sines or cosines of arcs, multiple of $\varphi$, and also cosines $pt + \varepsilon$. After that oscillation equations are transformed into a system of linear equations with constant coefficients, determining u, v, and w in relationship to x. Let us establish these equations, but first let us examine the order of this system. Expressions $\varkappa_1$, $\varkappa_2$, and $\tau$ include only second derivatives, expressions $\varepsilon_1$, $\varepsilon_2$, and $\omega$ include first derivatives. Thus, $M_1$ and $M_2$ contain second derivatives, but $N_1$ — third derivatives. The third equation (1.2), consequently, contains $\frac{\partial^4 \omega}{\partial x^4}$ in those terms, which are lowered, when an equation of oscillations with elongations is formed. Thus, full equations will be of a significantly higher order than equation of oscillations with elongations, the first ones will be of the eighth order, the second ones — of the fourth order. Lowering of order upon the transformation from the full system to equations of oscillations with elongations has a fundamental value and in general, is quite independent of the special cylindrical shape of the middle surface.

In accordance with the above, we shall write that:

$$\begin{aligned} u &= U \sin n\varphi \cos(pt+\varepsilon), \\ v &= V \cos n\varphi \cos(pt+\varepsilon), \\ w &= W \sin n\varphi \cos(pt+\varepsilon), \end{aligned} \tag{2.1}$$

where U, V, W are functions of x. It follows from this that

$$\begin{aligned} \varepsilon_1 &= \frac{dU}{dx} \sin n\varphi \cos(pt+\varepsilon), \\ \varepsilon_2 &= -\frac{W+nV}{a} \sin n\varphi \cos(pt+\varepsilon), \\ \omega &= \left(\frac{\partial V}{dx} + n\frac{U}{a}\right) \cos n\varphi \cos(pt+\varepsilon); \\ \varkappa_1 &= \frac{d^2W}{dx^2} \sin n\varphi \cos(pt+\varepsilon), \\ \varkappa_2 &= -\frac{nV+n^2W}{a^2} \sin n\varphi \cos(pt+\varepsilon), \\ \tau &= \frac{1}{a}\frac{d}{dx}(V+nW) \cos n\varphi \cos(pt+\varepsilon), \end{aligned} \tag{2.2}$$

and, consequently,

$$\begin{aligned} M_1 &= -D \sin n\varphi \cos(pt+\varepsilon)\left(\frac{d^2W}{dx^2} - \nu\frac{nV+n^2W}{a^2}\right), \\ M_2 &= -D \sin n\varphi \cos(pt+\varepsilon)\left(\nu\frac{d^2W}{dx^2} - \frac{nV+n^2W}{a^2}\right), \\ M_{12} &= D \cos n\varphi \cos(pt+\varepsilon)\frac{1-\nu}{a}\left(n\frac{dW}{dx} + \frac{dV}{dx}\right) = -M_{21}. \end{aligned}$$

The first two equations from (1.1) of this chapter will have the form,

$$\begin{aligned} N_1 &= \frac{\partial M_1}{\partial x} + \frac{1}{a}\frac{\partial M_{21}}{\partial \varphi}, \\ N_2 &= \frac{1}{a}\frac{\partial M_2}{\partial \varphi} - \frac{\partial M_{12}}{\partial x}, \end{aligned}$$

or, which is the same:

$$\begin{aligned} N_1 &= -D \sin n\varphi \cos(pt+\varepsilon)\left\{\frac{d^3W}{dx^3} - \frac{1}{a^2}\left(n^2\frac{dW}{dx} + n\frac{dV}{dx}\right)\right\}, \\ N_2 &= -D \cos n\varphi \cos(pt+\varepsilon)\left\{\frac{n}{a}\frac{d^2W}{dx^2} - \frac{n^3}{a^3}W + \frac{1-\nu}{a}\frac{d^2V}{dx^2} - \right. \\ &\qquad \left. - \frac{n^2}{a^3}V\right\}. \end{aligned}$$

Further we have:

$$T_1 = D\left[\frac{3}{h^2}(\varepsilon_1 + \nu\varepsilon_2) + \frac{2 - 2\nu - 3\nu^2}{2(1-\nu)}\frac{\varkappa_1}{a} - \frac{2\nu - \nu^2}{2(1-\nu)}\frac{\varkappa_2}{a}\right],$$

$$T_2 = D\left[\frac{3}{h^2}(\varepsilon_2 + \nu\varepsilon_1) - \frac{\nu + 2\nu^2}{2(1-\nu)}\frac{\varkappa_1}{a} - \frac{2+\nu}{2(1-\nu)}\frac{\varkappa_2}{a}\right],$$

$$S_1 = \frac{1}{2}D(1-\nu)\left[\frac{3}{h^2}\omega + \frac{\tau}{a}\right],$$

$$S_2 = \frac{1}{2}D(1-\nu)\left[-\frac{3}{h^2}\omega + \frac{\tau}{a}\right],$$

where $\varepsilon_1$, ..., $\varkappa_1$, ..., have values, given by formulas (2.2). The oscillation equation will be,

$$\frac{\partial T_1}{\partial x} - \frac{1}{a}\frac{\partial S_2}{\partial \varphi} + 2\rho h p^2 u = 0,$$

$$\frac{\partial S_1}{\partial x} + \frac{1}{a}\frac{\partial T_2}{\partial \varphi} - \frac{N_2}{a} + 2\rho h p^2 v = 0,$$

$$\frac{\partial N_1}{\partial x} + \frac{1}{a}\frac{\partial N_2}{\partial \varphi} + \frac{T_2}{a} + 2\rho h p^2 w = 0,$$

or, introducing U, V. and W, we will obtain:

$$\frac{3D}{h^2}\left[\frac{d}{dx}\left(\frac{dU}{dx} - \nu\frac{W+nV}{a}\right) - \frac{1-\nu}{2}\frac{n}{a}\left(\frac{dV}{dx} + \frac{nU}{a}\right)\right] + 2\rho p^2 U +$$
$$+ \frac{D}{h}\left[\frac{2-2\nu-3\nu^2}{2(1-\nu)a}\frac{d^3W}{dx^3} + \frac{1+2\nu^2}{2(1-\nu)}\frac{n}{a^3}\frac{d}{dx}(V+nW)\right] = 0, \quad (2.3)$$

$$\frac{3D}{h^2}\left[\frac{n}{a}\left(\nu\frac{dU}{dx} - \frac{W+nV}{a}\right) + \frac{1-\nu}{2}\frac{d}{dx}\left(\frac{dV}{dx} + \frac{n}{a}U\right)\right] + 2\rho p^2 V +$$
$$+ \frac{D}{h}\left[-\frac{\nu+2\nu^2}{2(1-\nu)}\frac{n}{a^2}\frac{d^2W}{dx^2} + \frac{2+\nu}{2(1-\nu)}\frac{n^2}{a^4}(V+nW) +\right.$$
$$+ \frac{1}{2}\frac{1-\nu}{a^2}\frac{d^2}{dx^2}(V+nW) + \frac{n}{a^2}\frac{d^2W}{dx^2} - \frac{n^3}{a^4}W +$$
$$\left. + \frac{1-\nu}{a^2}\frac{d^2V}{dx^2} - \frac{n^2}{a^4}V\right] = 0; \quad (2.4)$$

$$\frac{3D}{h^2}\left[\frac{\nu}{a}\frac{dU}{dx} - \frac{W+nV}{a^2}\right] + 2\rho p^2 W - \frac{D}{h}\left[\frac{d^4W}{dx^4} - \frac{2n^2}{a^2}\frac{d^2W}{dx^2} +\right.$$
$$+ \frac{n^4}{a^4}W - (2-\nu)\frac{n}{a^2}\frac{d^2V}{dx^2} + \frac{n^3}{a^4}V + \frac{\nu+2\nu^2}{2(1-\nu)a^2}\frac{d^2W}{dx^2} -$$
$$\left. - \frac{2+\nu}{2(1-\nu)}\frac{n}{a^4}(nV+W)\right] = 0. \quad (2.5)$$

Conditions on the boundary with $x = \pm l$ will be as follows:

$$T_1 = 0, \quad S_1 + \frac{M_{12}}{a} = 0, \quad N_1 - \frac{1}{a}\frac{\partial M_{12}}{\partial \varphi} = 0, \quad M_1 = 0,$$

where all the left-hand parts of equalities are linear functions of U, V, and W and their derivatives with respect to x.

The system of equations for determination of u, v, and w depending upon x constitutes a linear system with constant coefficients of the eighth order. This system contains an unknown value $p^2$, and also known values h and n, where n is an arbitrarily selected number of wave lengths, which can be laid along the length of the circumference. Let us assume that u, v, and w in addition to factors, containing $\varphi$ and t, are proportional to values $\xi e^{mx}$, $\eta e^{mx}$, $\zeta e^{mx}$, where $\xi$, $\eta$, $\zeta$, and m are constants. Constant m will be the root of the equation, obtained by means of equating the determinant to zero; this will be an equation of the eighth degree with respect to m or the fourth degree with respect to $m^2$ and will not contain terms with odd powers of m. The coefficients of this equation depend on $p^2$. If m will satisfy this equation, then relationships $\xi:\eta:\zeta$ will be determined depending upon m and $p^2$ from any two equations of motion. Not taking into consideration the factors, depending on $\varphi$ and t, we can write:

$$u = \sum_{i=1}^{4} (\xi_i e^{m_i x} + \xi_i' e^{-m_i x}),$$
$$v = \sum_{i=1}^{4} (\eta_i e^{m_i x} + \eta_i' e^{-m_i x}), \qquad (2.6)$$
$$w = \sum_{i=1}^{4} (\zeta_i e^{m_i x} + \zeta_i' e^{-m_i x}),$$

where $\xi_i$ and $\xi_i'$ are arbitrary constants, and $\eta_i$, ..., will be proportional to the first two constants. Boundary conditions $x = \pm l$

yield eight linear homogeneous equations with respect to $\xi_i$ and $\xi_i'$. Exclusion of these constants leads to one equation for determination of $p^2$; this will be the equation of frequencies.

Let us examine longitudinal oscillations (oscillation of elongations). Equations of longitudinal oscillations are obtained by means of rejecting in equations (2.3), (2.4), (2.5) terms, containing factor $\frac{D}{h}$. The equation for determination of $m^2$ becomes a quadratic equation. Conditions when $x = \pm l$ are reduced to equalities:

$$T_1 = 0, \quad S_1 = 0,$$

or

$$\frac{dU}{dx} - \nu \frac{W + nV}{a} = 0,$$
$$\frac{dV}{dx} + \frac{n}{a} U = 0.$$

Since h is not included in these equations, then the frequency will not depend on h.

Under the condition of symmetrical oscillations, when u, v, and w do not depend on φ, we have:

$$u = U \cos(pt + \varepsilon), \quad v = V \cos(pt + \varepsilon), \quad w = W \cos(pt + \varepsilon);$$

substituting in oscillation equations, we will obtain:

$$\begin{gathered}
\frac{E}{1-\nu}\left(\frac{d^2U}{dx^2} - \frac{\nu}{a}\frac{dW}{dx}\right) + \rho p^2 U = 0, \\
\frac{E}{2(1+\nu)}\frac{d^2V}{dx^2} + \rho p^2 V = 0, \\
\frac{E}{1-\nu^2}\left(\frac{\nu}{a}\frac{dU}{dx} - \frac{1}{a^2}W\right) + \rho p^2 W = 0.
\end{gathered} \tag{2.7}$$

Boundary conditions with $x = \pm l$ will be,

$$\frac{dU}{dx} - \nu\frac{W}{a} = 0, \quad \frac{dV}{dx} = 0. \tag{2.8}$$

There exists two kinds of symmetrical oscillations. In the first U and W disappear, so that the displacement will be tangential to the normal section of the cylinder. In this case we have,

$$V = \eta \cos \frac{n\pi x}{l}, \quad p^2 = \frac{E}{2\rho(1+\nu)} \frac{n^2\pi^2}{l^2}, \tag{2.9}$$

where n is an integer. In the second kind of oscillations V disappears so that displacements occur in the plane, passing through the axis; here:

$$U = \xi \cos \frac{n\pi x}{l}, \quad W = \zeta \sin \frac{n\pi x}{l}, \tag{2.10}$$

where $\xi$ and $\zeta$ are enterconnected by equations,

$$\left[p^2 - \frac{E}{\rho(1-\nu^2)} \frac{n^2\pi^2}{l^2}\right]\xi - \frac{E\nu}{\rho(1-\nu^2)} \frac{n\pi}{la}\zeta = 0,$$
$$\left[p^2 - \frac{E}{\rho(1-\nu^2)} \frac{1}{a^2}\right]\zeta - \frac{E\nu}{\rho(1-\nu^2)} \frac{n\pi}{la}\xi = 0. \tag{2.11}$$

The equation for the frequencies will be

$$p^4 - p^2 \frac{E}{\rho(1-\nu^2)}\left(\frac{1}{a^2} + \frac{n^2\pi^2}{l^2}\right) + \frac{E^2 n^2\pi^2}{\rho^2(1-\nu^2)a^2 l^2} = 0. \tag{2.12}$$

If the length of the cylinder is great in comparison with its diameter, i.e., $\frac{a}{l}$ is small, then there are two types of oscillations, 1) almost purely radial with frequency

$$\frac{\left[\frac{E}{\rho(1-\nu^2)}\right]^{\frac{1}{2}}}{2\pi a},$$

2) and almost purely longitudinal with frequency

$$\frac{n\left(\frac{E}{\rho}\right)^{\frac{1}{2}}}{2l}.$$

The latter are similar to longitudinal oscillations of a thin rod (oscillation of elongations).

Let us now examine oscillation without elongations. Such oscillations along the generatrix are determined by formulas:

$$u = 0, \quad v = A_n \cos(p_n t + \varepsilon_n) \cos(n\varphi + \alpha_n),$$
$$w = -nA_n \cos(pt + \varepsilon_n) \sin(n\varphi + \alpha_n). \qquad (2.13)$$

where

$$p_n^2 = \frac{D}{2\rho h a^4} \frac{n^2(n^2-1)^2}{n^2+1}. \qquad (2.14)$$

If oscillations occur in three dimensions, displacements will have the following form:

$$u = -\frac{a}{n} B_n \cos(p'_n t + \varepsilon'_n) \sin(n\varphi + \beta_n),$$
$$v = xB_n \cos(p'_n t + \varepsilon'_n) \cos(n\varphi + \beta_n), \qquad (2.15)$$
$$w = -nxB_n \cos(p'_n t + \varepsilon'_n) \sin(n\varphi + \beta_n).$$

where

$$p_n'^2 = \frac{D}{2\rho h a^4} \frac{n^2(n^2-1)}{n^2+1} \frac{1 + \frac{6(1-\nu)a^2}{n^2 l^2}}{1 + \frac{3a^2}{n^2(n^2+1) l^2}}. \qquad (2.16)$$

As we can see, values p and p' here are proportional to h.

In the latter case, when we introduce the assumption of the possibility of oscillations, not accompanied by elongations, an inaccuracy is admitted, owing to which the equations of motion, and boundary conditions are not fully satisfied. Besides, it turns out that in order to satisfy different equations, it is necessary to introduce a correction which contains small changes due to displacement, while to satisfy boundary conditions the correction for displacement should be more significant than the one, which is necessary to satisfy differential equations.

Let us clarify the character of the corrections, which must be introduced into the deformation without elongations. The existence of oscillations, not accompanied by elongations is connected with the fact that the order of the system of motion equations is lowered from eight (oscillation with elongations) to four. In the frequency equation (in the case of oscillation with elongations) terms,

containing $m^8$ and $m^6$, have factor $h^2$, and, thus, two values of $m^2$ will be large numbers of the order $\frac{1}{h}$. In order to show, how with the help of the solution, depending on large values of m, it would be possible to satisfy conditions on the boundary, we will examine Lamb's example [7].

A cylindrical shell, limited by two generatrices and two circumferences of normal sections, is subjected to action of forces, applied along the generatrices (circumferences are free from forces); it is distorted, turning into a surface of revolution, in such a way that the displacement, tangential to the circumference of normal section v is proportional to φ. Let us find this displacement.

We have $v = c\varphi$, where c is a constant and displacements u and w do not depend on φ. Hence:

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \frac{c-w}{a}, \quad \omega = 0, \quad \varkappa_1 = \frac{\partial^2 w}{\partial x^2}, \quad \varkappa_2 = \frac{c}{a^2}, \quad \tau = 0.$$

Forces $S_1$, $S_2$ and moments $M_{12}$, $M_{21}$ disappear, and $M_1$, $M_2$, $N_1$, $N_2$ will be equal:

$$M_1 = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\nu c}{a^2}\right), M_2 = -D\left(\frac{c}{a^2} + \nu\frac{\partial^2 w}{\partial x^2}\right),$$

$$N_1 = -D\frac{\partial^3 w}{\partial x^3}, \quad N_2 = 0.$$

Equations of equilibrium will assume the form:

$$\frac{\partial T_1}{\partial x} = 0, \quad \frac{\partial T_2}{\partial \varphi} = 0, \quad -D\frac{\partial^4 w}{\partial x^4} + \frac{T_2}{a} = 0,$$

and the condition when $x = \pm l$ will lead to equalities:

$$T_1 = 0, \quad N_1 = 0, \quad M_1 = 0.$$

In order to satisfy these equations and conditions, we assume that $\varepsilon_1$ and $\varepsilon_2$ will be values of the same order as $h\varkappa_1$ and $h\varkappa_2$. If this takes place, then the forces can be expressed, with a sufficiently close approximation, in the following manner:

If n > 1, then two normal oscillations of the second class correspond to each n, and the lowest tone corresponds to the slowest of the two oscillations of this class when n = 2. Its frequency will be

$$p=\frac{1}{a}\sqrt{\frac{G}{\rho}}(1.176). \tag{3.3}$$

if Poisson's ratio $\nu = 0.25$, the frequencies of all these oscillations do not depend on the thickness of the shell.

In the specific case of the plane plate oscillations are divided into two main classes: one of them corresponds to deformations without elongations with displacements, which are normal to the plane of the plate; the second – to deformations, accompanied by elongations, when displacements are parallel to the plane of the plate. Here we can have longitudinal oscillations, when displacements are parallel to the plane of the plate; oscillations of this class are divided into two subclasses, the first subclass includes such oscillations, in which the middle plane does not undergo deformation; the second includes oscillations, in which displacements are analogous to the tangent displacements in a closed thin spherical shell. Oscillations of the second class, with which displacement has both the normal component to the plane of the plate and the component, lying in this plane are also possible; if the plate is thin, the first component will be smaller than the second. The normal component of displacement disappears on the middle plane, and the normal component of rotation disappears everywhere, so that these oscillations are analogous to oscillations of the second class in a closed thin spherical shell. There is, further, still another class of bending oscillations, when the displacement has a normal and a tangential component, where the latter is smaller than the normal one in the case, when the plate is thin. The tangential component disappears on the middle plane, so

that the deformation can be approximately considered not to have any elongation. With these oscillations linear elements, which at the beginning were normal to the middle plane, during the entire movement remain rectilinear and normal to the same plane. The frequency of the oscillation is approximately proportional to the thickness of the plate. Similar oscillations without elongation, as noted earlier (see §2 of this chapter), in a closed spherical shell are impossible.

Among these extreme cases there is an open sphere or a spherical arch (dome). If the hole is small and the shell is almost closed, then its oscillation closely approaches the oscillations of the closed shell. If however, the solid angle, under which the hole in the shell is seen from a pole, located on a part of the sphere, locking the shell is small, and the radius of the sphere is large, then oscillations approach those of the plane plate. In intermediate cases we will find oscillations, which for all practical purposes belong either to the type of the oscillations which proceed without elongations or to the type of oscillations with elongations.

Investigation of oscillations without elongations of thin spherical shell with the boundary contour in the form of a circumference was performed for the first time by Rayleigh [8]: he applied the energy method. In the case of a hemisphere the frequency of the lowest pitch is equal to

$$p=\left(\frac{h}{a^2}\right)\sqrt{\frac{G}{\rho}}\,(4{,}279). \tag{3.4}$$

When angle $\alpha$ which determines the size of the hole approaches $\pi$, the sphere will be almost closed and the frequency of the lowest tone of these oscillations will be equal to

$$p=\frac{h^2}{a^2(\pi-\alpha)^2}\sqrt{\frac{G}{\rho}}\,(5{,}657). \tag{3.5}$$

Let us assume that angle $(\pi - \alpha)$ becomes sufficiently small; leaving h constant, we can obtain for the frequency of the lowest pitch of oscillations without elongations, a value larger by any amount than the lowest frequency of oscillations of a closed spherical shell (the oscillations of the latter, of course, will be with elongations). Thus, in the case of an almost closed shell the principal argument, with the help of which we verify the existence of oscillations, which have practically elongations, becomes superfluous.

When fundamental equations of oscillations are set up by the method, which is shown in §2 of this chapter for the cylindrical shell, we take the displacement components in a form, containing two factors, the first is the sine or cosine of an arc, which is a multiple of $\varphi$, the second constitutes elementary harmonious function of t; after that, equations are reduced to a linear system of the eighth order, which serves us to determine the dependency of displacement components on width $\theta$. Conditions on the free edges are expressed by equating to zero, for a specific value of $\theta$, certain linear expressions, connecting displacement components and their derivatives with respect to $\theta$. The order of the system is sufficient to enable us to satisfy these conditions. If the solution of the system of equations subordinate to boundary conditions, was found, this would lead to determination of the type of oscillations and their frequency.

Oscillations of elongations are investigated by the method, which is expounded in the problem on the cylindrical shell. The system of equations in this case will be of the fourth order, besides it will be necessary to satisfy two boundary conditions. With any form of oscillations, movement is composed of two motions; in the first, the radial component of displacement is absent; in the second, the

radial rotation component. Each of these motions is expressed with the help of the spherical function, but the order of the latter in general, will not be a whole number. The order of the spherical function, expressing the oscillation without radial displacement, is connected with the frequency by relationship (3.1) which has $\alpha$ instead of n; order $\beta$ of the spherical function, expressing displacement, when the radial component of rotation equals zero, is connected with frequency by relationship (3.2), in which n is replaced by $\beta$. Both $\alpha$ and $\beta$ are connected by a transcendental relationship, which constitutes an equation of frequencies. Oscillations are not divided into classes, as in the case of the closed shell; as the shape of the open shell approaches the shape of the closed shell, its oscillations of elongations are transformed into analogous oscillations for the closed shell.

The existence of oscillations, practically approaching oscillations without elongations, obviously, are intimately connected with the fact that upon assuming the presence of elongation oscillations we lower the order of the system of motion equations from the eighth to the fourth. As in the case of the cylindrical shell, it is possible to show that oscillations cannot be entirely unaccompanied by deformations of elongations and that the correction, necessary to satisfy the conditions on the edges, is greater than that, which is needed to satisfy motion equations. Hence it may be concluded, that on the free edge elongations are comparable in value with bending strains and that to all purposes these elongations are limited only by a narrow band near the edges.

If we imagine the gradual changes in the character of oscillations, appearing with the growth of curvature, starting with the plane plate

and finishing with the closed spherical shell, the class of oscillations, which proceed practically without elongations, will disappear completely. The basis for this should be sought in the rapid growth of the frequency of all oscillations belonging to this class, upon a significant decrease of the hole in the shell.

## §4. Asymptotic Method of Investigation of Oscillations

For investigation of natural oscillations of plates, sloping shells, and also unsloping shells during oscillations with high indices of changeability of shape, V. V. Bolotin proposed an effective asymptotic method [9-11], the essence of which in the general formulation can be presented thus.

In a certain retangular (in a generalized sense) spatial region of variables $x_1, x_2, \ldots, x_m$, $(0 \leq x_1 \leq a_1,\ i = 1, 2, \ldots, m)$ we seek functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, satisfying the system of differential equations

$$\sum_{\alpha=1}^{n} L_{j\alpha}(\varphi_\alpha) - \lambda \sum_{\alpha=1}^{n} M_{j\alpha}(\varphi_\alpha) = 0 \qquad (j = 1, 2, \ldots, n) \tag{4.1}$$

and, on every border of the region, satisfying conditions:

$$N_{i\alpha}(\varphi_1, \varphi_2, \ldots, \varphi_n/0) = 0, \quad N_{i\alpha}(\varphi_1, \varphi_2, \ldots, \varphi_n/a_i) = 0 \qquad (i = 1, 2, \ldots, n;\ \alpha = 1, 2, \ldots, r). \tag{4.2}$$

Where $L_{j\alpha}$, $M_{j\alpha}$, and $N_{i\alpha}$ are linear differential operators, and 2r is the general order of the system (4.1). It is assumed that the boundary value problem is self-conjugate. It is necessary to determine for this problem the eigenvalues of $\lambda$ and eigenfunctions of $\varphi_j$.

According to the author [11], let us introduce the classification of the border self-conjugate problem. The problem, the eigenfunctions of which permit their presentation in the form of the product of

functions, depending only on each of the arguments individually:

$$\varphi_j = \prod_{i=1}^{m} \varphi_{ji}(x_i) \qquad (j = 1, 2, \ldots, n), \tag{4.3}$$

we will call a boundary value problem with separable variables. Solution of such problems, as a rule, is sought in the form of the product of trigonometric functions.

We shall term the self-conjugate problem (4.1)-(4.2) a boundary value problem with quasiseparating variables, if either the system (4.1) permits a generating solution in the form (4.3) owing to the corresponding selection of conditions (4.2), or system (4.1) permits a solution in the form

$$\varphi_j = \Phi_{ji_0}(x_{i_0}) \prod_{i \neq i_0}^{m} \varphi_{ji}(x_i) \qquad (i_0 = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n), \tag{4.4}$$

where $\Phi_{ji_0}$ are certain functions of one variable $x_{i_0}$. Consequently, substitution (4.4) transforms (4.1) into a system of differential equations with respect to functions $\Phi_{ji_0}$; or substitution (4.4) in conditions (4.2), which correspond to $i = i_0$, transforms them into conditions, containing only $\Phi_{ji_0}$.

An example of boundary-value problems with quasiseparating variables are problems, described by systems of differential equations with constant coefficients, which contain derivatives of even orders; boundary condition for every boundary must contain an operation of differentiation with respect to every "transverse" coordinate of the same parity.

V. V. Bolotin's asymptotic method for boundary-value problems

with quasidividing variables is based* on the assumption of possibility of the use of solution (4.3)** as a generating solution also for those problems, which do not permit an exact solution in this form; in addition we should consider it only as an approximate solution for an internal region, sufficiently removed from the boundaries. Near the boundaries the exact solution will differ from the generating solution; this phenomenon is termed by dynamic edge effect. Here if for every boundary we will manage to construct a solution, satisfying all the conditions in it and tending to a generating solution as it is removed further from the boundary, then asymptotic expressions for eigenfunctions can be found by means of a "gluing" operation.

In setting up the solution one should bear in mind every function $\psi_{ji}(x_i)$ in (4.3) in the corresponding selection of conditions for normalization contains two constants: the wave number $k_i$ and certain phase response. These constants can be found only after "gluing" of solutions. After substitution of (4.3) in (4.1) we will find a bond between eigenvalue $\lambda$ and wave numbers $k_1, k_2, \ldots, k_m$:

$$\lambda = \lambda(k_1, k_2, \ldots, k_m). \tag{4.5}$$

Actually, we will examine, for instance, boundary $x_{i_0} = 0$. Assuming that parameter $\lambda$ in equations (4.1) is determined according to (4.5), we will look for their solution in the form (4.4). A system of ordinary differential equations obtained in such a way will have, as it was already noted earlier, a solution of type $\psi_{ji_0}(x_{i_0})$

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*Only qualitative considerations are adduced here.

**Solution (4.3) possesses asymptotic properties of eigenfunctions, which are maintained also when boundary conditions are changed.

which depends on two arbitrary constants. If this system, furthermore, admits $r - 1$ linearly independent solutions, possessing properties of the boundary effect, then we will have $r + 1$ arbitrary constants. And, consequently, we will satisfy $r$ conditions on the boundary and the normalization condition. Analogously we will construct a solution also for the opposite boundary $x_{i_0} = a_{i_0}$. Requiring that in the internal region both solutions coincide with the accurecy of the solution of the edge effect type, we will obtain condition of "gluing", containing as unknowns wave numbers $k_1, k_2, \ldots, k_m$. In all we may obtain $m$ such conditions, after which eigenvalues are determined by for formula (4.5).

It is possible to expect that the error of operation of "gluing" has an order of values, which is adopted by functions of dynamic edge effect in the internal region. Consequently the faster the edge effect damps the smaller is this error.

The investigation of solutions shows that with the growth of wave numbers the error decreases rapidly. However, in certain instances, when for a certain region of wave numbers the solution of the type of edge effect, in general, cannot be constructed. In such cases, according to the author, we will discuss the degeneration of dynamic edge effect resulting from the strong influence of the boundary on the behavior of eigenfunctions in the internal region, in these cases the asymptotic method becomes invalid and, consequently, cannot be used.

Solutions, obtained by means of the asymptotic method,* can be considered to be approximate expressions for eigenfunctions, which

*Differential equations and boundary conditions are satisfied exactly here, however singleness of solution is attained only by the "gluing" operation, which needs a mathematical foundation, as also, does the entire method.

can be used everywhere, except in the vicinity of angular points of the region. It would be a sound practice to examine separately the solutions for the internal region and the solution for every boundary, as we do when we examine separately the moment and zero-moment solutions in the shell statics.

Let us apply V. V. Bolotin's asymptotic method to do research on natural oscillations of the sloping shell [11] and the circular cylindrical shell, using Yu. V. Gavrilov's resolution [12].

First, we will examine the natural oscillation of the sloping shell of constant thickness with radii of curvature $R_1$ = const and $R_2$ = const, supported on a rectangular frame [11]. Equations on sag w and tangential forces $\varphi$ are recorded thus:

$$D\nabla^2\nabla^2 w - \frac{1}{R_2}\frac{\partial^2\varphi}{\partial x_1^2} - \frac{1}{R_1}\frac{\partial^2\varphi}{\partial x_2^2} - \rho h p^2 w = 0,$$
$$\frac{1}{Eh}\nabla^2\nabla^2\varphi + \frac{1}{R_2}\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2 w}{\partial x_2^2} = 0. \tag{4.6}$$

The generating solution has the form,

$$w = \sin k_1(x_1 - x_1^0)\sin k_2(x_2 - x_2^0),$$
$$\varphi = \frac{Eh}{(k_1^2 + k_2^2)^2}\left(\frac{k_1^2}{R_2} + \frac{k_2^2}{R_1}\right)\sin k_1(x_1 - x_1^0)\sin k_2(x_2 - x_2^0), \tag{4.7}$$

where $k_1$ and $k_2$ are wave numbers, $x_1^0$ and $x_2^0$ are phase responses (limit phases).

Solution for edge $x_1 = 0$ we take in the form of:

$$w = W_1(x_1)\sin k_2(x_2 - x_2^0),$$
$$\varphi = \Phi_1(x_1)\sin k_2(x_2 - x_2^0). \tag{4.8}$$

Substitution (4.8) in (4.6) leads to a system of ordinary differential equations with constant coefficients with respect to $W_1(x_1)$ and $\Phi_1(x_1)$. The dynamic edge effect does not degenerate, if the characteristic equation of this system, set up with the additional condition that

$$p^2 = \frac{D}{\rho h}\left[(k_1^2 + k_2^2)^2 + \frac{Eh}{D}\frac{\left(\frac{k_1^2}{R_2} + \frac{k_2^2}{R_1}\right)^2}{(k_1^2 + k_2^2)^2}\right], \tag{4.9}$$

has three roots with negative real parts. Let us note, that the edge effect does not degenerate with any values of $k_1$ and $k_2$, if $R_1 \geq R_2$ (it also never degenerates for plates). With $R_1 < R_2$ degeneration is possible only with sufficiently small $k_1$ and $k_2$. For instance, for the asymptotic edge in a shell of zero Gaussian curvature the edge effect degenerates, if

$$k_1^2+k_2^2<\sqrt{\frac{Eh}{DR_1^2}}.$$

"Gluing" together the solutions, which originate at the boundaries $x_1 = 0$ and $x_1 = a_1$ and at boundaries $x_2 = 0$ and $x_2 = a_2$, we will obtain two equations for finding $k_1$ and $k_2$. After that frequency p is found by the formula (4.9). Let us note that with large wave numbers $k_1 \approx \frac{m_1\pi}{a_1}$ and $k_2 \approx \frac{m_2\pi}{a_2}$, where $m_1$ and $m_2$ are positive integers and formula (4.9) is turned into an estimate of the type of well-known appraisals of Courant-Weyl.*

Now let us investigate the spectrum of natural oscillations of circular cylindrical shells [12].

Let us combine with the middle surface of the shell an orthogonal curvilinear system of coordinates $x_1$, $x_2$ so that line $x_2$ = const coincide with the generatrices.

Forms of oscillations for normal sag $w_*$ are determined from the resolving equation

$$\nabla^2\nabla^2\nabla^2\nabla^2 w_* + \frac{Eh}{DR^2}\frac{\partial^4 w_*}{\partial x_1^4} - \frac{\gamma h p^2}{gD}\nabla^2\nabla^2 w_* = 0. \tag{4.10}$$

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*With large wave numbers the equations of the classical theory of shells (and plates) become insuitable and must be replaced by equations, which take into account deformation of the shift and rotation inertia. The asymptotic method may be applied to this class of problems also.

Here $\gamma$ is the specific gravity of the shell material, g is the acceleration of the force of gravity.

Solution of equation (4.10) near the circular edge is in the form

$$w_0 = W_1(x_1)\sin k_2(x_2 - x_2^0)$$

and analogously near the rectilinear edge

$$w_0 = W_2(x_2)\sin k_1(x_1 - x_1^0).$$

Function $W_2(x_2)$ is determined by expression

$$W_2(x_2) = C_{21}\sin k_2x_2 + C_{22}\cos k_2x_2 + \sum_{j=1}^{3} C_{2(j+2)}e^{s_{2j}x_2}. \tag{4.11}$$

Here $s_{2j}$ are negative roots of the characteristic euqation for edge effect near the rectilinear edge. Function $W_1(x_1)$ has the form

$$W_1(x_1) = C_{11}\sin k_1x_1 + C_{12}\cos k_1x_1 + \sum_{j=1}^{3} C_{1(j+2)}e^{s_{1j}x_1}, \tag{4.12}$$

if all roots of the corresponding characteristic equation assume real values, and

$$W_1(x_1) = C_{11}\sin k_1x_1 + C_{12}\cos k_1x_1 + C_{13}e^{s_{11}x_1} + \\ + C_{14}\sin\beta x_1e^{\alpha x_1} + C_{15}\cos\beta x_1e^{\alpha x_1}, \tag{4.13}$$

if roots $s_{12}$ and $s_{13}$ are complexly conjugated, $s_{12,\,13} = \alpha \pm i\beta$.

The first two terms in expressions (4.11), (4.12) and (4.13) correspond to the asymptotic expression for forms of oscillations, but the three remaining ones describe the dynamic edge effect.

Values of $s_{2j}$ are expressed in the explicit form, but $s_{1j}$ can be represented graphically or determined from the characteristic equation directly.

Wave numbers $k_1$ and $k_2$ are determined from conditions of "gluing" together, which, for cylindrical panel, in the case of identical conditions on opposite edges are recorded in the form:

$$\frac{k_1 a_1}{\pi} = n_1 + \frac{2}{\pi} \operatorname{arctg}\left(-\frac{C_{12}}{C_{11}}\right),$$
$$\frac{k_2 a_2}{\pi} = n_2 + \frac{2}{\pi} \operatorname{arctg}\left(-\frac{C_{22}}{C_{21}}\right). \tag{4.14}$$

where $n_1$, $n_2$ = 1, 2, 3, ...; $a_1$ and $a_2$ are sides of panel.

After determination of wave numbers it is easy to calculate the frequency of oscillations,

$$p^2 = \frac{gD}{\gamma h}\left[(k_1^2 + k_2^2)^2 + \frac{Eh}{DR^2}\frac{k_1^4}{(k_1^2 + k_2^2)^2}\right].$$

Let us now determine the parameters of the dynamic edge effect near the circular edge. Let us examine the edge effect near the fastened circular edge and near the circular edge with sliding fastening.

In the first case boundary conditions have the form

$$w_* = \frac{\partial w_*}{\partial x_1} = u_* = v_* = 0 \quad \text{when } x_1 = 0 \tag{4.15}$$

and in the second

$$w_* = \frac{\partial w_*}{\partial x_1} = \frac{\partial^2 \varphi_*}{\partial x_2^2} = \frac{\partial^2 \varphi_*}{\partial x_1 \partial x_2} = 0 \text{ when } x_1 = 0. \tag{4.16}$$

Here $u_*$ and $v_*$ are tangential displacements, which can be determined from relationships,

$$\nabla^2\nabla^2\left(\frac{\partial u_*}{\partial x_1}\right) = \frac{1}{R}\frac{\partial^2}{\partial x_1^2}\left[\nabla^2 w_* - (1+\nu)\frac{\partial^2 w_*}{\partial x_1^2}\right],$$
$$\nabla^2\nabla^2\left(\frac{\partial v_*}{\partial x_2}\right) = -\frac{1}{R}\frac{\partial^2}{\partial x_2^2}\left[\nabla^2 w_* + (1+\nu)\frac{\partial^2 w_*}{\partial x_1^2}\right], \tag{4.17}$$

where $\varphi_*$ is a function of efforts in the middle surface, for the determination of which we can use the second Mushtari - Vlasov equation [13]. Then

$$\nabla^2\nabla^2\varphi_* = -\frac{Eh}{R}\frac{\partial^2 w_*}{\partial x_1^2}. \tag{4.18}$$

From (4.17) and (4.18) we find,

$$u_* = U_1(x_1)\sin k_2(x_2 - x_2^0),$$
$$v_* = V_1(x_1)\cos k_2(x_2 - x_2^0),$$
$$\varphi_* = \Phi_1(x_1)\sin k_2(x_2 - x_2^0),$$

where in the case of nonoscillating edge effect,

$$U_1 = \frac{k_1}{R}K(C_{11}\cos k_1x_1 - C_{12}\sin k_1x_1) - \sum_{j=1}^{3} C_{1(j+2)}\frac{s_{1j}}{R}S_{1j}e^{s_{1j}x_1},$$

$$V_1 = \frac{k_2}{R}\left[K^*(C_{11}\sin k_1x_1 + C_{12}\cos k_1x_1) - \sum_{j=1}^{3} C_{1(j+2)}S^*_{1j}e^{s_{1j}x_1}\right],$$

$$\Phi_1 = \frac{Eh}{R}\left[K^{**}(C_{11}\sin k_1x_1 + C_{12}\cos k_1x_1) - \sum_{j=1}^{3} C_{1(j+2)}S^{**}_{1j}e^{s_{1j}x_1}\right].$$

Here we introduce designations

$$K = \frac{\nu k_1^2 - k_2^2}{(k_1^2 + k_2^2)^2}, \qquad S_{1j} = \frac{\nu s_{1j}^2 + k_2^2}{(s_{1j}^2 - k_2^2)^2},$$
$$K^* = \frac{(2+\nu)k_1^2 + k_2^2}{(k_1^2 + k_2^2)^2}, \qquad S^*_{1j} = \frac{(2+\nu)s_{1j}^2 - k_2^2}{(s_{1j}^2 - k_2^2)^2},$$
$$K^{**} = \frac{k_1^2}{(k_1^2 + k_2^2)^2}, \qquad S^{**}_{1j} = \frac{s_{1j}^2}{(s_{1j}^2 - k_2^2)^2}.$$

Further, assuming that $C_{11} = 1$, from (4.15) and (4.16) we find constants $C_{1k}$ ($k = 2, 3, 4, 5$). For determination of wave numbers we will be interested only in ratios $\frac{C_{12}}{C_{11}}$, which for a clamped circular edge are equal to:

$$\frac{C_{12}}{C_{11}} = k_1\frac{\sum_{j=1}^{3} P_{1j}}{\sum_{j=1}^{3} P_{2j}} \tag{4.19}$$

and for edge with sliding fastening

$$\frac{C_{12}}{C_{11}} = k_1\frac{\sum_{j=1}^{3} P_{3j}}{\sum_{j=1}^{3} P_{4j}}. \tag{4.20}$$

Here,

$$P_{11} = s_{11}(\dot{S}_{12} - \dot{S}_{13})(K + S_{11}).$$
$$P_{21} = s_{11}s_{12}(S_{11} - S_{12})(K^{\bullet} + \dot{S}_{13}).$$
$$P_{31} = s_{11}(\dot{S}_{12} - \ddot{S}_{13})(K^{\bullet\bullet} + \ddot{S}_{11}).$$
$$P_{41} = s_{11}s_{12}(\ddot{S}_{11} - \ddot{S}_{12})(K^{\bullet\bullet} + \ddot{S}_{13}).$$

and expressions for $P_{kj}$ (k = 1, 2, 3, 4; j = 1, 2, 3) are obtained from $P_{kj}$ by cyclic permutation of second indices.

In the case of the oscillating edge effect the corresponding expressions are obtained analogously.

With the help of (4.19) and (4.20) from equations (4.14) we calculate wave numbers for closed cylindrical shells with rigid and sliding fastening at the circular edge, as well as for panels with the same conditions at the circular edge and with free support at the rectilinear edge.

The solution of system (4.14) presents certain difficulties. However, the first approximation may be obtained by the graphic method relatively simply.

After the wave numbers are determined, oscillation frequencies are easily calculated. Furthermore, calculating all roots of characteristic equations $s_{1j}$ and $s_{2j}$ (j = 1, 2, 3) and all constants $C_{1k}$ and $C_{2k}$ (k = 1, 2, 3, 4, 5), we obtain expressions for function of forces, as well as for forms of oscillations and for bending moments and forces. Thus we can determine the stresses near the edges.

The asymptotic method may be applied also to problems on forced oscillations which is based on making use of expansion into series with respect to asymptotic expressions for forms of natural oscillations. This method can be successfully applied for the analysis of vibrations of plates and shells during high-frequency excitation.

Since in this case simplifications are possible, because with large wave numbers the influence of the edge effect is localized in narrow regions near the lines of distortion. Consequently, it is permissible to resort to expansion with respect to eigenfunctions of the generating solution (under the condition that phase responses are found, taking into account boundary conditions). After all coefficients of decompositions are found, we can calculate bending moments, severing forces, stresses and so forth in the dynamic edge effect zone.

## §5. Parametric Oscillations. Formulation of Problem

In preceding paragraphs we examined natural oscillations, when an oscillating body is isolated from any external influences. Such oscillations appear after an external action, which determines the initial deflection and initial speed, i.e., initial conditions, but the latter simply determine the subsequent process in the elastic system. An elastic system itself from the point of view of natural oscillations, in general, is determined by two parameters, which characterize the oscillatory process: natural frequency p and decrement δ (or damping coefficient).

Forced oscillations in the body, i.e., oscillation under the action of external forces, are determined not only by the physical properties of the body and by the parameters of the elastic system (p and δ), but also by external forces; mathematically this is expressed by the fact that into the equation a term enters which, depends explicitly on time. However an external influence of another form is possible, when an external force does not act on a body, and at the same time either parameters of the system (included in the coefficients of the equation) depend on time, or the external influence changes the parameters of the elastic system. Appearance of an

oscillatory process due to variation of parameters is called parametric excitation of oscillations, and oscillations are called parametric.

Thus, parametric oscillations are oscillations, appearing in an elastic system as the result of periodic change of those of its properties, which remain constant during free oscillations. And consequently, among oscillations, appearing in the presence of external periodic influence, we must distinguish two forms: forced oscillations and parametric oscillations. Forced oscillations are caused by the action of prescribed external forces on the elastic system, the properties of which are constant, i.e., the values (parameters), characterizing these properties, are constant. Parametric oscillations, on the contrary, appear owing to the periodic change of the elastic system itself.

The phenomenon of the build-up in time of the intensity of parametric oscillations of an elastic system is called parametric resonance. Parametric resonance appears with a definite relationship between the frequency of change of a parameter during external influence on the body and frequencies of its natural oscillations; it can arise every time, when the ratio

$$\zeta = \frac{\text{average natural frequency}}{\text{frequency of change of parameter}}$$

is close to one of the following values, $\frac{1}{2}$, 1, 2, 3, .... The condition for appearance of parametric resonance is fulfilled easier the larger the magnitude of the change of parameter, the less the loss of energy in the elastic system (friction or resistance), and the less the value $\zeta$. Therefore, it is observed most frequently when $\zeta = \frac{1}{2}$. The essential specific feature of parametric resonance is the

fact that it can appear in the presence of even an insignificant initial deflection of the elastic system from the state of equilibrium. In actual practice such deflections are always possible.

Let us outline the formulation of the problem on parametric oscillations of shells.*

Let us investigate the behavior of a shell under the influence of external surface load, variable in time according to periodic law:

$$X_0(\alpha, \beta, t), \quad Y_0(\alpha, \beta, t), \quad Z_0(\alpha, \beta, t). \tag{5.1}$$

Let us assume that load (5.1) induces in the shell a momentless state of strain and let us assume that in this state displacements of points of middle surfaces are equal to $u_0$, $v_0$, $w_0$. A change to the moment state will produce transpositions:

$$u = u_0 + \tilde{u}, \quad v = v_0 + \tilde{v}, \quad w = w_0 + \tilde{w}, \tag{5.2}$$

satisfying equations of the moment theory. Components X, Y, Z of the surface load consist of the reduced external load (5.1), forces of inertia and an additional induced load, appearing upon deflection of the middle surface from the initial momentless state:

$$\begin{aligned} X &= X_0 + \Delta X - m\frac{\partial^2 u}{\partial t^2}, \\ Y &= Y_0 + \Delta Y - m\frac{\partial^2 v}{\partial t^2}, \\ Z &= Z_0 + \Delta Z - m\frac{\partial^2 w}{\partial t^2}. \end{aligned} \tag{5.3}$$

Here m is the mass of the shell, referred to a unit of area of the middle surface.

Introducing (5.2) and (5.3) in equations of the moment theory in V. Z. Vlasov's [15] form and taking into account that undisturbed parameters are connected by equations,

*Here and in subsequent paragraphs of this chapter V. V. Bolotin's [14] results are expounded.

$$L_{11}(u_0)+L_{12}(v_0)+L_{13}(w_0)+\frac{1-\nu^2}{Eh}\left(X_0-m\frac{\partial^2 u_0}{\partial t^2}\right)=0,$$
$$L_{21}(u_0)+L_{22}(v_0)+L_{23}(w_0)+\frac{1-\nu^2}{Eh}\left(Y_0-m\frac{\partial^2 v_0}{\partial t^2}\right)=0,$$
$$L_{31}(u_0)+L_{32}(v_0)+L_{33}(w_0)+\frac{1-\nu^2}{Eh}\left(Z_0-m\frac{\partial^2 w_0}{\partial t^2}\right)=0.$$

we obtain "equations in variations."

$$\begin{aligned}
&L_{11}(\tilde u)+L_{12}(\tilde v)+L_{13}(\tilde w)+\frac{1-\nu^2}{Eh}\left(\Delta X-m\frac{\partial^2 \tilde u}{\partial t^2}\right)=0,\\
&L_{21}(\tilde u)+L_{22}(\tilde v)+L_{23}(\tilde w)+\frac{1-\nu^2}{Eh}\left(\Delta Y-m\frac{\partial^2 \tilde v}{\partial t^2}\right)=0,\\
&L_{31}(\tilde u)+L_{32}(\tilde v)+L_{33}(\tilde w)+\frac{1-\nu^2}{Eh}\left(\Delta Z-m\frac{\partial^2 \tilde w}{\partial t^2}\right)=0,
\end{aligned} \tag{5.4}$$

bars above $\tilde u$, $\tilde v$, $\tilde w$ are subsequently omitted. Here $L_{11}$, $L_{12}$, ..., are linear di ferential operators, referred to lines of the main curvatures: h is the thickness of the shell.

Regarding determination of components of a reduced load $\Delta X$, $\Delta Y$, and $\Delta Z$, it may be carried out in the following manner. ;t us assume that the zero-moment state is characterized by normal forces $T_1$ $(\alpha, \beta, t)$ and $T_2$ $(\alpha, \beta, t)$, which will be considered positive, if they cause compression. Disregarding forces of inertia of the zero-moment state, we can calculate the internal forces from equations of equilibrium of the shell element in this state:

$$\begin{aligned}
&\frac{\partial}{\partial \alpha}(BT_1)-T_2\frac{\partial B}{\partial \alpha}=ABX_0,\\
&\frac{\partial}{\partial \beta}(AT_2)-T_1\frac{\partial A}{\partial \beta}=ABY_0,\\
&k_1T_1+k_2T_2=Z_0,
\end{aligned} \tag{5.5}$$

where $k_1$, $k_2$ are the main curvatures.

Let us assume that, as it was earlier, $\varepsilon_1$ and $\varepsilon_2$ are relative longitudinal deformations, $\varkappa_1$ and $\varkappa_2$ is the increase of main curvatures due to moment deformation. The first quadratic form assumes the form

$$ds^2 = A^2(1+\varepsilon_1)^2 d\alpha^2 + B^2(1+\varepsilon_2)^2 d\beta^2.$$

Now, if in equations (5.5) instead of coefficients of the first quadratic form A and B we introduce $A(1 + \varepsilon_2)$ and $B(1 + \varepsilon_2)$ respectively; furthermore, if in the last equation we replace $k_1$ and $k_2$ by $k_1 + \varkappa_1$ and $k_2 + \varkappa_2$, then in this case they are not satisfied identically and, consequently, it is necessary instead of $X_0$, $Y_0$, $Z_0$ to take $X_0 + \Delta X$, $Y_0 + \Delta Y$, $Z_0 + \Delta Z$, where $\Delta X$, $\Delta Y$, and $\Delta Z$ are the additional (reduced) load. Thus, equation (5.5) should be written in the following form:

$$\frac{\partial}{\partial\alpha}[B(1+\varepsilon_2)T_1] - T_2\frac{\partial}{\partial\alpha}[B(1+\varepsilon_2)] =$$
$$= AB(1+\varepsilon_1)(1+\varepsilon_2)(X_0+\Delta X),$$
$$\frac{\partial}{\partial\beta}[A(1+\varepsilon_1)T_2] - T_1\frac{\partial}{\partial\beta}[A(1+\varepsilon_1)] =$$
$$= AB(1+\varepsilon_1)(1+\varepsilon_2)(Y_0+\Delta Y),$$
$$(k_1+\varkappa_1)T_1 + (k_2+\varkappa_2)T_2 = Z_0 + \Delta Z.$$

Taking into account (5.5) and disregarding values of the second order of smallness (products of type $\varepsilon_1\varepsilon_2$, $\varepsilon_1\,\Delta X$), we arrive at the following formulas:

$$\Delta X = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(\varepsilon_2 BT_1) - T_2\frac{\partial}{\partial\alpha}(\varepsilon_2 B)\right] - X_0(\varepsilon_1+\varepsilon_2),$$
$$\Delta Y = \frac{1}{AB}\left[\frac{\partial}{\partial\beta}(\varepsilon_1 AT_2) - T_1\frac{\partial}{\partial\beta}(\varepsilon_1 A)\right] - Y_0(\varepsilon_1+\varepsilon_2),$$
$$\Delta Z = T_1\varkappa_1 + T_2\varkappa_2. \qquad (5.6)$$

Let us introduce in formulas (5.6) instead of $\varepsilon_1$, $\varepsilon_2$, $\varkappa_1$, $\varkappa_2$ expressions:

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}v + k_1 w,$$
$$\varepsilon_2 = \frac{1}{AB}\frac{\partial B}{\partial\alpha}u + \frac{1}{B}\frac{\partial v}{\partial\beta} + k_2 w, \qquad (5.7)$$
$$\varkappa_1 = \frac{\partial k_1}{\partial\alpha}\frac{u}{A} + \frac{\partial k_1}{\partial\beta}\frac{v}{B} - k_1^2 w -$$
$$-\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta},$$
$$\varkappa_2 = \frac{\partial k_2}{\partial\alpha}\frac{u}{A} + \frac{\partial k_2}{\partial\beta}\frac{v}{B} - k_2^2 w -$$
$$-\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}.$$

Now it is easy to note that the terms in equations (5.4), containing $\Delta X$, $\Delta Y$, $\Delta Z$, are linear with respect to internal forces $T_1$ and $T_2$, and also with respect to displacements u, v, w and their derivatives. In the case of a periodic external load, of forces $T_1$ and $T_2$, are also periodic functions of time; system (5.4) in this case has periodic coefficients. Assuming that

$$\begin{aligned} u(\alpha,\beta,t) &= \sum u_k(t)\varphi_k(\alpha,\beta), \\ v(\alpha,\beta,t) &= \sum v_k(t)\psi_k(\alpha,\beta), \\ w(\alpha,\beta,t) &= \sum w_k(t)\chi_k(\alpha,\beta), \end{aligned} \tag{5.8}$$

where functions are selected so that corresponding boundary conditions are satisfied, and putting them in (5.4), we will reduce the problem to a system of ordinary differential equations with periodic coefficients. Methods of solution of such are sufficiently well developed.

## §6. The Closed Cylindrical Shell

Let us assume that a circular cylindrical shell with a radius of the middle surface R and thickness h is loaded with an evenly distributed radial load $q_0 + qt \cos \theta t$ and is compressed by a longitudinal force $P_0 + P_t \cos \theta t$. We shall use the system of coordinates in accordance with Fig. 5, introducing a dimensionless longitudinal coordinate $\alpha = \frac{z}{R}$. We will designate the displacement in the direction of the generatrix by u, the circumferential displacements by v and the radial displacement by w.

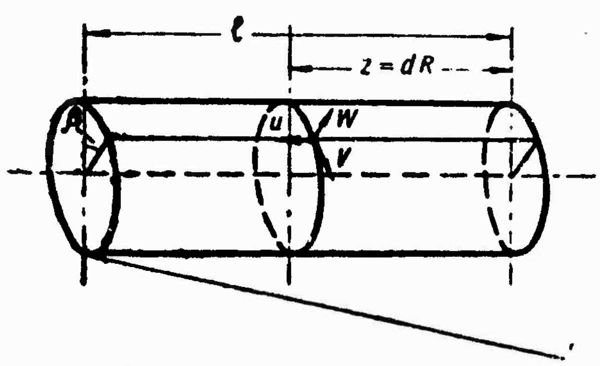


Fig. 5.

In equations (5.4) of this chapter, within tolerance limits assumed with respect to a cylindrical shell, one should write:

$$L_{11}=\frac{\partial^2}{\partial\alpha^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\beta^2},\quad L_{22}=\frac{\partial^2}{\partial\beta^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\alpha^2},$$
$$L_{12}=L_{21}=\frac{1+\nu}{2}\frac{\partial^2}{\partial\alpha\,\partial\beta},\quad L_{13}=L_{31}=\nu\frac{\partial}{\partial\alpha},\tag{6.1}$$
$$L_{23}=L_{32}=\frac{\partial}{\partial\beta},\quad L_{33}=c^2\nabla^2\nabla^2.$$

Here,

$$c^2=\frac{h^2}{12R^2},\quad \nabla^2=\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2},$$
$$\nabla^2\nabla^2=\nabla^4=\frac{\partial^4}{\partial\alpha^4}+2\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+\frac{\partial^4}{\partial\beta^4}.\tag{6.2}$$

Noting that internal forces corresponding to the initial zero-moment state, are expressed in the form:

$$T_1=\frac{1}{2\pi R}(P_0+P_t\cos\theta t),$$
$$T_2=R(q_0+q_t\cos\theta t),\tag{6.3}$$

and that in the case under consideration $A = B = R$, $k_1 = 0$, $k_2 = \frac{1}{R}$ and according to the formula (5.7) of this chapter,

$$\varepsilon_1=\frac{1}{R}\frac{\partial u}{\partial\alpha},\quad \varepsilon_2=\frac{1}{R}\left(\frac{\partial v}{\partial\beta}+w\right),$$
$$\varkappa_1=-\frac{1}{R^2}\frac{\partial^2 w}{\partial\alpha^2},\quad \varkappa_2=-\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\beta^2}+w\right),$$

it is not difficult to write the expressions of components of the reduced load $\Delta X$, $\Delta Y$, $\Delta Z$ using formulas (5.6):

$$\Delta X=\frac{T_1-T_2}{R^2}\frac{\partial}{\partial\alpha}\left(\frac{\partial v}{\partial\beta}+w\right),$$
$$\Delta Y=-\frac{T_1-T_2}{R^2}\frac{\partial^2 u}{\partial\alpha\,\partial\beta},\tag{6.4}$$
$$\Delta Z=-\frac{1}{R^2}\left[T_1\frac{\partial^2 w}{\partial\alpha^2}+T_2\left(\frac{\partial^2 w}{\partial\beta^2}+w\right)\right].$$

Thus, for a cylindrical shell equations (5.4) of this chapter assume the form:

$$\begin{gathered}
\frac{\partial^2 u}{\partial\alpha^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial\beta^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial\alpha\,\partial\beta}+\nu\frac{\partial w}{\partial\alpha}+\\
+\frac{1-\nu^2}{Eh}\left[\frac{T_1-T_2}{R^2}\frac{\partial}{\partial\alpha}\left(\frac{\partial v}{\partial\beta}+w\right)-m\frac{\partial^2 u}{\partial t^2}\right]=0,\\
\frac{1+\nu}{2}\frac{\partial^2 u}{\partial\alpha\,\partial\beta}+\frac{\partial^2 v}{\partial\beta^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial\alpha^2}+\frac{\partial w}{\partial\beta}-\\
-\frac{1-\nu^2}{Eh}\left[\frac{T_1-T_2}{R^2}\frac{\partial^2 u}{\partial\alpha\,\partial\beta}+m\frac{\partial^2 v}{\partial t^2}\right]=0,\\
\nu\frac{\partial u}{\partial\alpha}+\frac{\partial v}{\partial\beta}+c^2\nabla^2\nabla^2 w+\frac{1-\nu^2}{Eh}\left\{\frac{1}{R^2}\left[T_1\frac{\partial^2 w}{\partial\alpha^2}+\right.\right.\\
\left.\left.+T_2\left(\frac{\partial^2 w}{\partial\beta^2}+w\right)\right]+m\frac{\partial^2 w}{\partial t^2}\right\}=0.
\end{gathered}\tag{6.5}$$

We seek the solution for the system of equations (6.5) in the form:

$$\begin{gathered}
u=U(t)\cos n\alpha\cos k\beta,\\
v=V(t)\sin n\alpha\sin k\beta,\\
w=W(t)\sin n\alpha\cos k\beta,
\end{gathered}\tag{6.6}$$

where $n = \frac{i\pi R}{l}$, where i and k become positive integers. Here i indicates the number of half-waves in the meridional direction ($l$ is the length of the shell), k indicates the number of half-waves in the circumferential direction. The solution in form (6.6) corresponds to the case, when on the ends of the shell ($z = 0$ and $z = l$) both the radial and the circumferential displacements ($u \neq 0$) disappear.

We can easily prove by direct substitution that equations (6.5) are identically satisfied, if functions U(t), V(t), W(t) are determined from a system of ordinary differential equations:

$$\begin{gathered}
\frac{m(1-\nu^2)}{Eh}\frac{d^2U}{dt^2}+\left(n^2+\frac{1-\nu}{2}k^2\right)U-\frac{1+\nu}{2}nkV-\\
-\nu nW-\frac{1-\nu^2}{Eh}\frac{T_1-T_2}{R^2}n(kV+W)=0,\\
\frac{m(1-\nu^2)}{Eh}\frac{d^2V}{dt^2}-\frac{1+\nu}{2}nkU+\left(k^2+\frac{1-\nu}{2}n^2\right)V+\\
+kW+\frac{1-\nu^2}{Eh}\frac{T_1-T_2}{R^2}nkU=0,\\
\frac{m(1-\nu^2)}{Eh}\frac{d^2W}{dt^2}-\nu nU+kV+c^2(n^2+k^2)^2W-\\
-\frac{1-\nu^2}{Eh}\frac{W}{R^2}[T_1n^2+T_2(k^2-1)]=0.
\end{gathered}\tag{6.7}$$

The system of equations (6.7) may be represented in the matrix form

$$m\frac{d^2f}{dt^2}+(R-T_1S_1-T_2S_2)f=0,$$

where f is the vector with components U, V, and W,

$$R=\frac{Eh}{1-\nu^2}\begin{Vmatrix} n^2+\frac{1-\nu}{2}k^2 & -\frac{1+\nu}{2}nk & -\nu n \\ -\frac{1+\nu}{2}nk & k^2+\frac{1-\nu}{2}n^2 & k \\ -\nu n & k & c^2(n^2+k^2)^2 \end{Vmatrix}.$$

$$S_1=\frac{1}{R^2}\begin{Vmatrix} 0 & nk & n \\ -nk & 0 & 0 \\ 0 & 0 & n^2 \end{Vmatrix}, \quad S_2=\frac{1}{R^2}\begin{Vmatrix} 0 & -nk & n \\ nk & 0 & 0 \\ 0 & 0 & k^2-1 \end{Vmatrix}.$$

Frequencies of natural oscillations of an unloaded shell are determined from equation

$$|R-p^2E|=0,$$

and critical parameters of longitudinal compressing, and radial loads are determined from equations

$$\left|R-\frac{P}{2\pi R}S_1\right|=0 \text{ и } |R-qRS_2|=0.$$

The problem of dynamic stability leads in the first approximation to the equation

$$\left|R-\frac{1}{2\pi R}\left(P_0\pm\frac{1}{2}P_t\right)S_1-\right.$$
$$\left.-\left(q_0\pm\frac{1}{2}q_t\right)RS_2-\frac{1}{4}\theta^2E\right|=0. \qquad (6.8)$$

In case it is possible to disregard the effect of tangential forces of inertia and tangential components of the reduced load, then the problem about oscillations of a cylindrical shell is reduced to one "resolving" equation. For a nonsloping cylindrical shell it has the form:

$$(\nabla^2+1)^2\nabla^2\nabla^2\Phi-(1-\nu)\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\beta^2}\right)\nabla^2\Phi+$$
$$+\frac{1-\nu^2}{c^2}\frac{\partial^4\Phi}{\partial\alpha^4}-\frac{R^4}{D}Z=0. \tag{6.9}$$

here

$$\nabla^2\nabla^2\Phi=w.$$

Using (6.9) and the last formula (6.4), we obtain,

$$(\nabla^2+1)^2\nabla^2\nabla^2\Phi-(1-\nu)\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\beta^2}\right)\nabla^2\Phi+$$
$$+\frac{1-\nu^2}{c^2}\frac{\partial^4\Phi}{\partial\alpha^4}+\frac{mR^4}{D}\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2\Phi+$$
$$+\frac{R^2}{D}\left[T_1\frac{\partial^2}{\partial\alpha^2}+T_2\left(\frac{\partial^2}{\partial\beta^2}+1\right)\right]\nabla^2\nabla^2\Phi=0. \tag{6.10}$$

Let us assume that $\lambda$ is the length of a half-wave in the meridional (or circumferential) direction. Then the first term in equation (6.10) will have the order $\sim\left(\frac{R}{\lambda}\right)^8$; the second term $\sim\left(\frac{R}{\lambda}\right)^6$; the third $\sim\frac{R^6}{\lambda^4h^2}$. If the length of a half-wave is small as compared to the radius, then the second component in (6.10) can be disregarded. Disregarding on the basis of similar considerations the other terms of the similar order of smallness we arrive at equation

$$\nabla^2\nabla^2\nabla^2\nabla^2\Phi+\frac{1-\nu^2}{c^2}\frac{\partial^4\Phi}{\partial\alpha^4}+\frac{R^4}{D}\left(\frac{T_1}{R^2}\frac{\partial^2}{\partial\alpha^2}+\frac{T_2}{R^2}\frac{\partial^2}{\partial\beta^2}+\right.$$
$$\left.+m\frac{\partial^2}{\partial t^2}\right)\nabla^2\nabla^2\Phi=0. \tag{6.11}$$

This equation completely corresponds to the known equation for mildly sloping shells.

Thus, returning to the general equation (6.10) and assuming that in it

$$\Phi(\alpha,\beta,t)=f(t)\sin n\alpha\cos k\beta,$$

which corresponds to case (6.6), we arrive at an ordinary differential equation

$$\frac{mR^4}{D}\frac{d^2f}{dt^2}+g(n,k)f-\frac{R^2}{D}[T_1n^2+T_2(k^2-1)]f=0, \tag{6.12}$$

where

$$g(n,k)=\frac{(n^2+k^2+1)^2(n^2+k^2)^2+(1-\nu)n^2(n^4-k^4)+\frac{1-\nu^2}{c^4}n^4}{(n^2+k^2)^2}. \tag{6.13}$$

Let us introduce designations:

$$\frac{Dg(n,k)}{mR^4}=p^2,\quad \frac{Dg(n,k)}{n^2R^2}=T_{1*},\quad \frac{Dg(n,k)}{(k^2-1)R^2}=T_{2*}. \tag{6.14}$$

Equation (6.12) may be now recorded in the form

$$\frac{d^2f}{dt^2}+p^2\left(1-\frac{P}{P_*}-\frac{q}{q_*}\right)f=0, \tag{6.15}$$

where $P_* = 2\pi RT_{1*}$, $q_* = \frac{T_{2*}}{R}$. Consequently, the problem is reduced to a known equation.

In conclusion let us note that these results are easily generalized for the case of the orthotropic cylindrical shell. Really, let us assume that $E_1$, $E_2$ and $\nu_1$, $\nu_2$ are moduli of elasticity and Poisson's ratio in directions $\beta$ = const and $\alpha$ = const respectively; G is the shear modulus. Let us introduce the following differential operators:

$$\begin{aligned}
\nabla_0^2 &= E_0\frac{\partial^2}{\partial\alpha^2}+E_2\frac{\partial^2}{\partial\beta^2},\\
\nabla_1^4 &= E_1\frac{\partial^4}{\partial\alpha^4}+E_4\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+E_2\frac{\partial^4}{\partial\beta^4},\\
\nabla_2^4 &= E_1\frac{\partial^4}{\partial\alpha^4}+E_3\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+E_2\frac{\partial^4}{\partial\beta^4},
\end{aligned} \tag{6.16}$$

where

$$\begin{aligned}
E_0 &= 2G(1-\nu_1\nu_2)+E_2\nu_1,\\
E_3 &= 4G(1-\nu_1\nu_2)+E_1\nu_2+E_2\nu_1,\\
E_4 &= \frac{E_1E_2}{G}-E_1\nu_2-E_2\nu_1.
\end{aligned} \tag{6.17}$$

The equation, analogous to (6.11), for the orthotropic shell assumes the following form

$$\begin{aligned}
&\nabla_1^4\nabla_2^4\Phi+\frac{E_1E_2(1-\nu_1\nu_2)}{c^4}\frac{\partial^4\Phi}{\partial\alpha^4}+\\
&+\frac{1-\nu_1\nu_2}{c^4h}\left(T_1\frac{\partial^2}{\partial\alpha^2}+T_2\frac{\partial^2}{\partial\beta^2}+mR^2\frac{\partial^2}{\partial t^2}\right)\nabla_1^4\Phi=0,
\end{aligned} \tag{6.18}$$

where $\Phi$ is the function, connected with the radial displacement by relationship (6.9).

Further transforms again result in equation (6.15), the coefficients of which for the case of a supported shell are determined without difficulty.*

## §7. The Spherical Shell

Let us investigate oscillations of the spherical shell under the action of a radial load evenly distributed on the surface:

$$Z_0 = -(q_0 + q_1 \cos\theta t). \tag{7.1}$$

Let us designate with R the radius of the middle surface, with h – the thickness, and adopt geographic coordinates $\psi$ and $\beta$ ($\psi$ is the angle of latitude, $\beta$ – angle of longitude, Fig. 6). Displacements of points of the middle surface will be disignated by u in the direction of line $\psi$ = const, by v in the direction of line $\beta$ = const, and by w – the radial displacement, positive in the direction of the external normal.

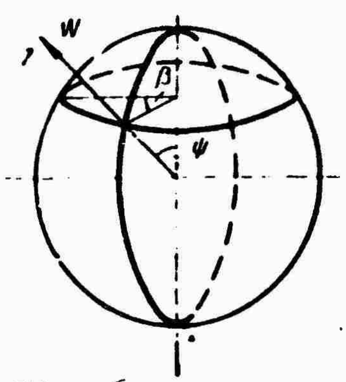

Fig. 6.

We know [15] that in the case of a radial load only, the system of equations of a spherical shell is reduced to one resolving equation

$$[c^2(\nabla^2+1)^2+1](\nabla^2+2)w = \frac{R^2 Z}{Eh}(\nabla^2 - 1 - \nu). \tag{7.2}$$

Here $\nabla^2(\ )$ is a Laplacian operator on the sphere

$$\nabla^2(\ ) = \frac{1}{\sin\psi}\left[\frac{\partial}{\partial\psi}\left(\sin\psi\frac{\partial}{\partial\psi}\right) + \frac{1}{\sin^2\psi}\frac{\partial^2}{\partial\beta^2}\right],$$
$$c^2 = \frac{h^2}{12(1-\nu^2)R^2}. \tag{7.3}$$

*See A. N. Markov [16].

In the zero-moment state the internal forces of the shell are reduced to compressing forces

$$T_1 = T_2 = \frac{1}{2} R (q_0 + q_t \cos \theta t). \qquad (7.4)$$

In the case of a spherical shell

$$k_1 = k_2 = \frac{1}{R},$$
$$A = R, \quad B = R \sin \psi,$$

and formulas (5.7) of this chapter assume the form:

$$\varkappa_1 = -\frac{1}{R^2}\left(w + \frac{\partial^2 w}{\partial \psi^2}\right),$$
$$\varkappa_2 = -\frac{1}{R^2}\left(w - \frac{1}{\sin^2 \psi}\frac{\partial^2 w}{\partial \beta^2} - \operatorname{ctg} \psi \frac{\partial w}{\partial \psi}\right).$$

Where, according to (5.6) of this chapter an additional given load, appearing upon deflection of the shell from the zero-moment state, will be

$$\Delta Z = \frac{1}{2} R (q_0 + q_t \cos \theta t)(\varkappa_1 + \varkappa_2),$$

or

$$\Delta Z = -\frac{1}{2R}(q_0 + q_t \cos \theta t)(\nabla^2 + 2) w. \qquad (7.5)$$

Components $\Delta X$ and $\Delta Y$ are equal to zero. In addition to the external pressure (7.1) and reduced load (7.5) the shell is acted upon by forces of inertia

$$-m\frac{\partial^2 u}{\partial t^2}, \quad -m\frac{\partial^2 v}{\partial t^2}, \quad -m\frac{\partial^2 w}{\partial t^2}. \qquad (7.6)$$

Rejecting, in accordance with the adopted approximation, the tangential components of the forces of inertia, we will find that forces, acting upon the shell, are reduced to a radial load

$$Z = -(q_0 + q_t \cos \theta t) - \frac{q}{2R}(q_0 + q_t \cos \theta t) \times$$
$$\times (\nabla^2 + 2) w - m\frac{\partial^2 w}{\partial t^2}. \qquad (7.7)$$

The first component produces a uniform compression of the shell and may be discarded if we understand that $w(\psi, \beta, t)$ is the deflection from an undisturbed, momentless state. Then equation (7.2) assumes the form

$$
\begin{gathered}
[c^2(\nabla^2+1)^2+1](\nabla^2+2)w+ \\
+\frac{(q_0+q_t\cos\theta t)R}{2Eh}(\nabla^2+1-\nu)(\nabla^2+2)w+ \\
+\frac{mR^2}{Eh}(\nabla^2+1-\nu)\frac{\partial^2 w}{\partial t^2}=0.
\end{gathered}
\tag{7.8}
$$

Let us look for the solution of equation (7.8) in the class of functions

$$w(\psi,\beta,t)=f(t)F(\psi,\beta). \tag{7.9}$$

where $f(t)$ is an unknown function of time, $F(\psi, \beta)$ are solutions of the differential equation

$$\nabla^2 F+\lambda F=0, \tag{7.10}$$

satisfying the boundary conditions for $w$ (i.e., conditions of continuity and the single-valuedness on the sphere). Substitution in (7.8) after reduction on $F(\psi, \beta)$ yields:

$$
\begin{gathered}
[c^2(\lambda-1)^2+1](\lambda-2)f-\frac{(q_0+q_t\cos\theta t)R}{2Eh}(\lambda-1+\nu)\times \\
\times(\lambda-2)f+\frac{mR^2}{Eh}(\lambda-1+\nu)\frac{d^2f}{dt^2}=0.
\end{gathered}
\tag{7.11}
$$

Let us introduce designations:

$$
\begin{gathered}
p^2=\frac{Eh}{mR^2}\,\frac{\lambda-2}{\lambda-1+\nu}[c^2(\lambda-1)^2+1], \\
q_* =\frac{2Eh}{R(\lambda-1+\nu)}[c^2(\lambda-1)^2+1]
\end{gathered}
\tag{7.12}
$$

and, introducing them in (7.11), we will obtain:

$$\frac{d^2f}{dt^2}+p^2\left(1-\frac{q_0+q_t\cos\theta t}{q_*}\right)f=0. \tag{7.13}$$

Formulas (7.12) give natural frequencies and critical forces, depending on the still unknown parameter $\lambda$. However, one practically important question may be solved to the end even without determination of $\lambda$. Boundaries of main regions of instability can be found by known approximate formulas. In particular, the lowest boundary

$$\theta_{*}^{2}=4p^{2}\left(1-\frac{q_{0}+\frac{1}{2}q_{t}}{q_{*}}\right). \tag{7.14}$$

For practical applications it is interesting to know the envelope of the lower bounds of regions of instability. Let us assume that parameter $\lambda$ can assume any real positive values, i.e., let us assume that equation (7.10) has a continuous spectrum of values. For a limited region, such as the sphere, spectrum of values is discrete. But in the vicinity of values of $\lambda$ which are of interest to us the spectrum of equation (7.10) is, nevertheless, sufficiently "thick," so that the error, following from the assumption made, is small.

Subsequently let us designate,

$$\theta_{*}^{2}=\frac{4Eh}{mR^{2}}q(\lambda), \tag{7.15}$$

where

$$q(\lambda)=(\lambda-2)\left[\frac{c^{2}(\lambda-1)^{2}+1}{\lambda-1+\nu}-\frac{\left(q_{0}+\frac{1}{2}q_{t}\right)R}{2Eh}\right]. \tag{7.16}$$

For determination of envelope let us assume that $\frac{dq}{d\lambda}=0$, hence, we will obtain an equation for $\lambda$. Let us consider the case of sufficiently large values of $\lambda \gg 1$. Then

$$q(\lambda)\approx c^{2}\lambda^{2}+1-\frac{\left(q_{0}+\frac{1}{2}q_{t}\right)R}{2Eh}\lambda, \tag{7.17}$$

and the root of equation $\frac{dq}{d\lambda} = 0$,

$$\lambda_* = \frac{\left(q_0 + \frac{1}{2} q_t\right) R}{4Ehc^2} \tag{7.18}$$

and consequently,

$$\theta_*^2 = \frac{4Eh}{mR^2}\left[1 - \frac{\left(q_0 + \frac{1}{2} q_t\right)^2 R^2}{16E^2h^2c^2}\right]. \tag{7.19}$$

Let us introduce designation:

$$\frac{4Ehc}{R} = q_{**}, \quad \frac{Eh}{mR^2} = p_0^2. \tag{7.20}$$

Formula (7.19) assumes the form

$$\theta_*^2 = 4p_0^2\left[1 - \frac{\left(q_0 + \frac{1}{2} q_t\right)^2}{q_{**}^2}\right]. \tag{7.21}$$

Certainly, one should take into account that $q_{**}$ constitutes an approximate (in the sense of the assumptions made) value of the minimum critical pressure. Actually, considering that

$$c^2 = \frac{h^2}{12R^2(1-\nu^2)},$$

we will obtain a well-known formula

$$q_{**} = \frac{2Eh^2}{R^2} \frac{1}{\sqrt{3(1-\nu^2)}}. \tag{7.22}$$

Finally, let us determine parameter $\lambda$ in the general case. Equation (7.10) on a sphere

$$\frac{1}{\sin\psi}\left[\frac{\partial}{\partial\psi}\left(\sin\psi \frac{\partial F}{\partial\psi}\right)\right] + \frac{1}{\sin^2\psi}\frac{\partial^2 F}{\partial\beta^2} + \lambda F = 0 \tag{7.23}$$

leads, as we know to spherical functions.

We find the solution of equation (7.23) in the form:

$$F(\psi, \beta) = P(\psi)\begin{matrix}\sin k\beta \\ \cos k\beta.\end{matrix} \tag{7.24}$$

The condition for single-valuedness of $F(\psi, \beta)$ on the sphere requires that k be an integer or a zero ($k = 0, 1, 2, \ldots$). Substitution in (7.23) yields:

$$\frac{1}{\sin\psi}\left[\frac{d}{d\psi}(\sin\psi)\frac{dP}{d\psi}\right]+\left(\lambda-\frac{k^2}{\sin^2\psi}\right)P=0 \quad (k=0, 1, 2, \ldots). \tag{7.25}$$

Assuming that $x = \cos\psi$, we reduce equation (7.25) to the form:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left(\lambda-\frac{k^2}{1-x^2}\right)P=0. \tag{7.26}$$

This is a known equation for associated Legendre polynomials. Equation (7.26) has eigenvalues

$$\lambda_n = n(n+1) \quad (n=0, 1, 2, \ldots). \tag{7.27}$$

Every eigenvalue $\lambda_n$ corresponds to (N + 1) eigenfunctions:

$$P_n^k(x)=(1-x^2)^{\frac{k}{2}}\frac{d^k}{dx^k}P_n(x), \quad (k=0, 1, 2, \ldots, n), \tag{7.28}$$

where

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n].$$

Now it is possible to record for equation (7.23) a system of its solutions:

$$\begin{aligned}
k=0 \quad & F_0(\psi,\beta)=P_n(\cos\psi), \\
k=1 \quad & F_{-1}(\psi,\beta)=P_n^{(1)}(\cos\psi)\sin\beta, \\
& F_1(\psi,\beta)=P_n^{(1)}(\cos\psi)\cos\beta, \\
& \quad \vdots \qquad\qquad \vdots \\
k=n \quad & F_{-n}(\psi,\beta)=P_n^{(n)}(\cos\psi)\sin n\beta, \\
& F_n(\psi,\beta)=P_n^{(n)}(\cos\psi)\cos n\beta.
\end{aligned} \tag{7.29}$$

We know that Legendre polynomials $P_n(x)$ have in the interval of change $\psi(0, \pi)$, exactly n zeroes. Associated functions $P_n^{(k)}(x)$ have accordingly (n - k) zeroes.

Since sin kβ and cos kβ turn into zero on 2k meridians, and $P_n^{(k)}(x)$ in view of that what was just now said – on (n - k) latitudes, then the entire sphere is partitioned into "cells" inside which $F(\psi, \beta)$ retains the constant sign. This means that number λ determines the form of the oscillation formula and, in particular, dimensions of "half-waves" in meridional and latitudinal directions. The smaller are the dimensions of half-waves, the larger is, consequently, the parameter λ. In this case the difference between two neighboring eigenvalues becomes small compared to their magnitude, which justifies the assumption about the continuity of change of λ.

## CHAPTER III

# FLUTTER OF PANELS AND SHELLS

## § 1. Formulation of the Problem

In the preceding chapter we investigated natural oscillations of parametric character. Other types of natural oscillations of plates and shells originate in their interaction with liquids and gas flows.

Here we examine a phenomenon, very interesting and essential for the technology of high speeds, which is termed panel flutter and consists of the fact that sheathing or other thin-walled elements of structures of the type of plates and shells, around which there is a supersonic fluid or gas flow, during specific critical speeds attain oscillatory motion with intensely growing amplitudes, which can bring the structure to destruction.

Theoretical research of panel flutter in a setting that is correct, in the physical and mathematical sense, became possible after the law of plane sections in the aerodynamics of large supersonic speeds [17] was established in 1947.

Analyzing the motion of thin solid bodies with great supersonic speeds in various media, A. A. Il'yushin discovered the following general property, which he termed the law of plane sections, "If the speed vector of any point of a body of a regular aerodynamic

form,* is V and if transverse speeds of its other points are of the order not exceeding $\varepsilon V$, then in either the established or transient motions the body produces in its environment only transverse perturbations, and the pressure at any point of the body surface, calculated according to this law, can differ from the true pressure by the value of the order not exceeding

$$\frac{1+e^2}{2M^2} = \frac{1}{2}\left(\varepsilon^2 + \frac{1}{M^2}\right)$$

as compared to unity." Here $M^2 = \frac{V^2}{v_0^2} \gg 1$ is Mach number, $e = \frac{\varepsilon V}{v_0}$ is Il'yushin's parameter,expressed through the body speed V, slope of the normal to area $\varepsilon$, and speed of sound in undisturbed medium $v_0$, i.e., speed of sound in gas at infinity; this parameter has fundamental value, inasmuch as both in linearized and nonelinearized theories in the presence of vortexes and shock waves the pressure on the body surface is determined only by these parameters and the form of the body.

Consequently, if before the body we separate by two neighboring parallel planes a layer of physical particles of the medium, perpendicular to the speed vector V of the body, then in calculating the pressure with the shown degree of accuracy, one may assume that particles of the medium will make motions, parallel to the planes, so that for them the plane would be like hard impenetrable walls.

The law of plane sections enabled us to give a new setting for supersonic aerodynamics problems (and the method of aerodynamic model studies); at the same time it made it possible to reduce the problem of

---

*This is the body, for which during the motion in a gas medium the normal to its surface deviates from the plane, perpendicular to the vector of speed V, by a small angle $\varepsilon$ in all points of the surface, with the exception of singular points or lines. However, inasmuch as in supersonic aerodynamics the state of flow in a certain cross section of the body depends only on the form of the front part of the body, all calculations, true for regular bodies, are also true for other thin bodies, having a regular front part, i.e., for plates and shells of various forms.

calculation for the established and transient motions to the simplest problem on the motion of a piston in a pipe of constant section,* where the piston moves according to the given law $v = v(t)$, and this is the speed, with which in a motionless column the surface cutting it, compresses the gas; for any point of the surface it is equal to the projection of the vector of absolute velocity of the surface element on the normal to this element.

Thus, it became possible to investigate theoretically in a correct and convenient for practical applications form the important problems, pertaining to motion of thin-walled structures in gas, determine pressures, and consequently all aerodynamic forces, acting on the supporting surface during great supersonic speeds, presence of shock waves and variable entropy of gas. The calculation is especially simple in the linearized theory,** in this case, for instance, over pressure $\Delta p$ on any area of the surface is equal to the pressure in motionless gas $p_0$, multiplied by politropy $\varkappa$ index, and the relation of normal component of the speed vector of this area $v(t)$ to the speed of sound in undisturbed gas $v_0$;

$$\Delta p = \varkappa p_0 \frac{v(t)}{v_0}.$$

In 1949 A. A. Il'yushin for the first time expressed the idea on the possibility of investigation of the panel flutter on the basis of these regularities and give a correct formulation of the problem.***

---

*This theory is true for $M > 1.5$ and small angles of attack $\varepsilon$.

**I. e. when with $M^2 \gg 1$ parameter $e < 1$ is nevertheless small at the expense either of the angle of incidence $\varepsilon$ or thickness of the profile of supporting surface and the gas entropy can be considered constant.

***The model of supporting surface in the form of a beam with a rigid chord, considered in the theory of bending-twisting flutter (M. V. Keldysh, Ye. P. Grossman, A. I. Nekrasov and others), is replaced by a model in the form of elastic plate and shell.

The first solution of the problem on plate flutter in this formulation took place in 1950 and belongs to A. A. Movchan; they also introduced the effective concept of the "stability parabola," which is widely used, and offered the method of obtaining exact solutions for the class of problems on rectangular plates, two sides of which, directed along the flow, are supported by hinges, and two remaining sides have arbitrary boundary conditions [18-22]. After him a number of authors — abroad and in our country have examined problems of this kind [23-28] as well as numerous other authors.

Problems on flutter as applied to shells for aerodynamic forces, considered in the form of overpressure [17], were studied by R. D. Stepanov; to him belongs the solution of problems on the flutter of cylindrical, spherical shells, and panels and the attempt of investigating plate flutter in nonlinear setting [29-31]. We know of research on critical speeds in nonlinear theory of aeroelasticity performed by V. V. Bolotin [32-33] and others.

Now we will give a formulation of the problem on the flutter of shells [34]. The principal layout of formulation of the problem, without lowering the general character of reasonings, can be sufficiently distinctly comprehended with the example of cylindrical shell flutter.

It is known that in the case, when a load, directed in every point along the normal to surface ($X = Y = 0$, $Z \neq 0$), acts on the shell, the basic solving equation for sloping cylindrical shells, without calculation of tangential forces of inertia, has the form [15]

$$\nabla^2\nabla^2\nabla^2\nabla^2\Phi + \frac{1-\nu^2}{c^2}\frac{\partial^4\Phi}{\partial x^4} = \frac{R^4}{D}Z. \tag{1.1}$$

Here $c^2 = \frac{h^2}{12R^2}$ is the constant, R — radius, $D = \frac{Eh^3}{12(1-\nu^2)}$ — cylindrical

rigidity, h — thickness, α, β are dimensionless coordinates of the point on cylindrical surface of the shell, constituting, α — distance along the generator expressed in fractions of radius R, β — central angle, $\nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}$, and, lastly, $\Phi(\alpha, \beta, t)$ — scalar function, determined by the formulas;

$$
\begin{aligned}
u &= \frac{\partial^3\Phi}{\partial\alpha\,\partial\beta^2} - \nu\frac{\partial^3\Phi}{\partial\alpha^3},\\
v &= -\left[\frac{\partial^3\Phi}{\partial\beta^3} + (2+\nu)\frac{\partial^3\Phi}{\partial\alpha^2\,\partial\beta}\right],\\
w &= \nabla^2\nabla^2\Phi = \nabla^4\Phi.
\end{aligned}
\tag{1.2}
$$

Internal forces are determined through function Φ with formulas;

$$
\begin{aligned}
&T_1 = \frac{Eh}{R}\frac{\partial^4\Phi}{\partial\alpha^2\,\partial\beta^2},\quad T_2 = \frac{Eh}{R}\frac{\partial^4\Phi}{\partial\alpha^4},\quad M_1 = \frac{D}{R^2}\left(\frac{\partial^2}{\partial\alpha^2} + \nu\frac{\partial^2}{\partial\beta^2}\right)\nabla^4\Phi,\\
&M_2 = \frac{D}{R^2}\left(\frac{\partial^2}{\partial\beta^2} + \nu\frac{\partial^2}{\partial\alpha^2}\right)\nabla^4\Phi,\\
&S = S_1 = -S_2 = -\frac{Eh}{R}\frac{\partial^4\Phi}{\partial\alpha^3\,\partial\beta},\\
&H = M_{12} = -M_{21} = -\frac{D(1-\nu)}{R^2}\frac{\partial^2}{\partial\alpha\,\partial\beta}\nabla^4\Phi,\\
&N_1 = -\frac{D}{R^3}\frac{\partial}{\partial\alpha}\nabla^6\Phi,\quad N_2 = -\frac{D}{R^3}\frac{\partial}{\partial\beta}\nabla^6\Phi.
\end{aligned}
\tag{1.3}
$$

Generalized transverse forces, determined in the Kirchhoff sense and necessary for the formulation of boundary conditions, are calculated by the formulas;

$$
\begin{aligned}
N_1^* &= -\frac{D}{R^3}\left[\frac{\partial^3}{\partial\alpha^3} + (2-\nu)\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right]\nabla^4\Phi,\\
N_2^* &= -\frac{D}{R^3}\left[\frac{\partial^3}{\partial\beta^3} + (2-\nu)\frac{\partial^3}{\partial\alpha^2\,\partial\beta}\right]\nabla^4\Phi.
\end{aligned}
\tag{1.4}
$$

In the system of dimensionless coordinates α, β when X = Y = 0, and Z ≠ 0 the basic solving equation for average-length* cylindrical shells has the form

$$
\frac{\partial^4\Phi_1}{\partial\alpha^4} + \frac{c^4}{1-\nu^2}\left[\frac{\partial^2}{\partial\beta^2} + 1\right]^2\frac{\partial^4\Phi_1}{\partial\beta^4} = \frac{R^2}{Eh}Z.
\tag{1.5}
$$

Here function $\Phi_1(\alpha, \beta, t)$ is determined by formulas,

---

*From now on we use the approximate theory of calculation of cylindrical shells of average length [35].

$$u = \frac{\partial^3 \Phi_1}{\partial\alpha\,\partial\beta^2}, \quad v = -\frac{\partial^3 \Phi_1}{\partial\beta^3}, \quad w = \frac{\partial^4 \Phi_1}{\partial\beta^4}. \tag{1.6}$$

Internal forces in this case through function $\Phi_1$ are expressed by relationships;

$$\begin{gathered}
T_1 = \frac{Eh}{R}\frac{\partial^4 \Phi_1}{\partial\alpha^2\,\partial\beta^2}, \quad T_2 = \frac{Eh}{R}\left[\frac{\partial^4 \Phi_1}{\partial\alpha^4} + \frac{c^2}{1-\nu^2}\left(\frac{\partial^8 \Phi_1}{\partial\beta^8} + \frac{\partial^4 \Phi_1}{\partial\alpha^4}\right)\right], \\
M_1 = \frac{D}{R^2}\nu\left[\frac{\partial^6 \Phi_1}{\partial\beta^6} + \frac{\partial^4 \Phi_1}{\partial\beta^4}\right], \quad M_2 = \frac{D}{R^2}\left[\frac{\partial^6 \Phi_1}{\partial\beta^6} + \frac{\partial^4 \Phi_1}{\partial\beta^4}\right], \\
S = -\frac{Eh}{R}\frac{\partial^4 \Phi_1}{\partial\alpha^3\,\partial\beta}, \quad H = -\frac{D(1-\nu)}{R^3}\left[\frac{\partial^6 \Phi_1}{\partial\alpha\,\partial\beta^5} + \frac{\partial^4 \Phi_1}{\partial\alpha\,\partial\beta^3}\right], \\
N_1 = -\frac{D}{R^3}\left[\frac{\partial^7 \Phi_1}{\partial\alpha\,\partial\beta^6} + \frac{\partial^5 \Phi_1}{\partial\alpha\,\partial\beta^4}\right], \quad N_2 = -\frac{D}{R^3}\left[\frac{\partial^7 \Phi_1}{\partial\beta^7} + \frac{\partial^5 \Phi_1}{\partial\beta^5}\right].
\end{gathered} \tag{1.7}$$

To differential equations (1.1) and (1.5) in every particular case we must add boundary conditions prescribed on edges of the shell.

Let us assume that the shell moves in the gas flow with constant speed V under the action of aerodynamic and other forces, originating from loads on the structure, and is in the state of relative equilibrium, which we call undisturbed equilibrium. Let us assume that $u^*(\alpha, \beta, t)$, $v^*$, $w^*$, $T_1^*(\alpha, \beta)$, ..., $\Phi^*(\alpha, \beta, t)$, $\Phi_1^*(\alpha, \beta, t)$ will be displacements and other corresponding functions in undisturbed motion. Then force $Z^*$, included in equation (1.1) or (1.5), according to [17] will be equal

$$Z^* = BV\frac{\partial w^*}{R\,\partial\alpha} - B_1\frac{\partial w^*}{\partial t}. \tag{1.8}$$

Here $B = \frac{p_0 \varkappa}{v_0}$ = const is the coefficient of swaying and $B_1$ = const — coefficient of damping, which reflect properties of the medium, in which the shell moves.

For greater generality it is of interest to study the boundary value problem for values V, included in the internal $0 \leq V \leq \infty$, which we propose to do subsequently.

Let us assume for states, other than stationary;

$$u = u^* + \tilde{u},\ v = v^* + \tilde{v},\ w = w^* + \tilde{w}, \dots\ \Phi = \Phi^* + \tilde{\Phi},$$
$$\Phi_1 = \Phi_1^* + \tilde{\Phi}_1.$$

Then force

$$Z = Z^* + \tilde{Z},$$

while on the basis of [17] and D'Alembert's principle

$$\tilde{Z} = BV \frac{\partial \tilde{w}}{R \partial \alpha} - B_1 \frac{\partial \tilde{w}}{\partial t} - \rho h \frac{\partial^2 w}{\partial t^2}, \tag{1.9}$$

where $\rho h \frac{\partial^2 \tilde{w}}{\partial t^2}$ is the force of inertia, $\rho$ — density of the shell material; and, consequently, equations (1.1) and (1.5), as well as the corresponding boundary conditions will become linear and uniform.

Equation (1.1) taking into account (1.2) assumes the form

$$\begin{gathered} C_*^2 \nabla^8 \Phi + \frac{\partial^4 \Phi}{\partial \alpha^4} + \frac{R^2}{E} \rho \frac{\partial^2}{\partial t^2} \nabla^4 \Phi - \frac{BVR}{Eh} \frac{\partial}{\partial \alpha} \nabla^4 \Phi + \\ + \frac{B_1 R^2}{Eh} \frac{\partial}{\partial t} \nabla^4 \Phi = 0. \end{gathered} \tag{1.10}$$

In its turn equation (1.5) taking into account (1.6) assumes this form*

$$\begin{gathered} \frac{\partial^4 \Phi_1}{\partial \alpha^4} + C_*^2 \left( \frac{\partial^2}{\partial \beta^2} + 1 \right)^2 \frac{\partial^4 \Phi_1}{\partial \beta^4} + \frac{R^2}{E} \rho \frac{\partial^6 \Phi_1}{\partial t^2 \partial \beta^4} - \\ - \frac{BVR}{Eh} \frac{\partial^5 \Phi_1}{\partial \alpha \partial \beta^4} + \frac{B_1 R^2}{Eh} \frac{\partial^5 \Phi_1}{\partial t \partial \beta^4} = 0. \end{gathered} \tag{1.11}$$

In equation (1.10) and (1.11) we introduced a new dimensionless value

$$C_*^2 = \frac{c^2}{1 - \nu^2} = \frac{h^2}{12 R^2 (1 - \nu^2)}. \tag{1.12}$$

Relationships (1.10) and (1.11) are equations of small oscillations of cylindrical shells. Together with corresponding boundary conditions they form the initial boundary-value problem of the shell flutter. This problem has the solution

$$\Phi(\alpha, \beta, t) \equiv 0. \tag{1.13}$$

---

*From (1.11) we can easily obtain an equation, describing plate flutter.

The flutter problem consists of clarification of conditions, under which the undisturbed motion, responding to trivial solution (1.13), is stable in the sense that the given smallness of perturbed motions at any moment of time $t \geq t_0$ will be guaranteed by the sufficient smallness of initial perturbations, given in the initial moment of time $t_0$.

Let us investigate the class of solutions of the type

$$\Phi(\alpha, \beta, t) = \Psi(\alpha, \beta) e^{\omega t}, \tag{1.14}$$

where $\omega = p + iq$ is the constant complex number (complex frequency), and $\psi(\alpha, \beta) = \psi_1(\alpha, \beta) + i\psi_2(\alpha, \beta)$ is a continuous together with eight derivatives, complex function of real values $\alpha$, $\beta$. It is obvious that in the class of solutions (1.14) condition $\mathrm{Re}\,\omega > 0$ will be a sufficient criterion of instability.

Let us give the name of critical speeds to those values of speed V, which separate regions of stable and unstable states of the shell.

The question of the relationship between the stability in class (1.14) and stability with respect to a broader class solutions of equations (1.10) and (1.11) is not considered here.

After introduction in equation (1.10) expression (1.14) instead of $\Phi$ and reduction by factor $e^{\omega t}$ we obtain for functions of $\psi(\alpha, \beta)$ the equation

$$C_0^2 \nabla^8 \psi + \frac{\partial^4 \psi}{\partial x^4} - \lambda \nabla^4 \psi - \frac{BVR}{Eh} \frac{\partial}{\partial x} \nabla^4 \psi = 0. \tag{1.15}$$

Here we assume that $B = B_1$ and introduce designation

$$-\lambda = \rho \frac{R^2 \omega^2}{E} + \frac{BR^3 \omega}{Eh}. \tag{1.16}$$

Adding here the given boundary conditions on the edges of the shell, we obtain the boundary-value problem, the solution of which gives values of $\lambda$ and eigenfunctions of $\psi(\alpha, \beta)$. From the relationship (1.16) we can easily find for every $\lambda$ two values of complex

frequency $\omega$;

$$\omega_{1,2}=-\frac{B}{2\rho h}\pm\left[\left(\frac{B}{2\rho h}\right)^2-\frac{E\lambda}{\rho R^2}\right]^{1/2}. \qquad (1.17)$$

In investigating the stability in class (1.14) those values of speed will be critical during transition through which the boundary-value problem acquires solutions of the form (1.14) with positive real part of the complex frequency $\omega$. One of the roots (1.17) always has a negative real part, because the sum of roots

$$\omega_1+\omega_2=-\frac{B}{2\rho h} \qquad (1.18)$$

is negative. Let us assume that for a certain $\lambda$ one of the roots of (1.17) is a purely imaginary number; $\mathrm{Re}\,\omega = p = 0$, $\omega = iq$. Then from (1.16) we find,

$$\mathrm{Re}\,\lambda=\lambda_1=\rho\frac{R^2}{E}q^2,\quad \mathrm{Im}\,\lambda=\lambda_2=-\frac{BR^2}{Eh}q. \qquad (1.19)$$

Equations (1.19) on the complex plane $\lambda_1$, $\lambda_2$ depict points of a square parabola (Fig. 7)

$$\lambda_1=\rho\frac{Eh^2}{B^2R^2}\lambda_2^2, \qquad (1.20)$$

which is called [18] parabola of stability. The region, lying inside the parabola of stability, corresponds to proper values, for which both roots of (1.17) have a negative real part, but the region, lying outside the parabola, corresponds to proper values, for which the real part of one of roots (1.17) is positive.

Thus, the problem of finding the critical speed in class (1.14) is reduced to the study of location of eigenvalues of $\lambda$ of the boundary-value problem (1.10) or (1.11) with respect to the stability parabola (1.20).

## § 2. Flutter of Panel

Let us assume that a slender body of aerodynamic shape moves in

a stationary gas rectilinearly and evenly at a high supersonic speed V. On the surface of the body we will examine a part of its sheathing — the rectangular panel, which in the undisturbed state, being plane, moves parallel to two own edges without an angle of incidence with respect to the gas* (Fig. 8). In the plane of this undisturbed motion of the panel we introduce a rectangular system of x and y coordinates, moving together with the body rectilinearly and evenly with speed V along axis x. Edges of panel at any moment of time t coincide with sectors of straight lines x = 0, x = a, y = 0, y = b.

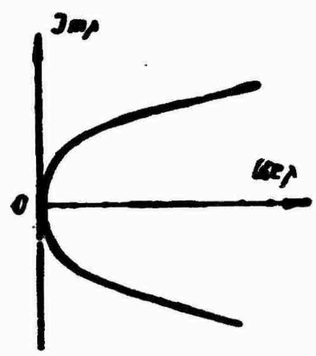

Fig. 7.

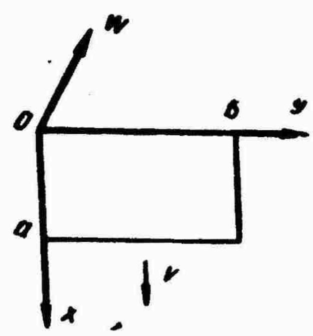


Fig. 8.

Under the influence of certain causes the undisturbed motion of the panel in its own plane may be disturbed, and the panel will begin to perform a perturbed motion with sag w(x, y, t), the positive value of which is determined by w axis in Fig. 8. Considering the panel to be thin and isotropic, we use for the description of its small sags w(x, y, t) the equation of bend of plate [3],

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\mu\frac{\partial^2 w}{\partial t^2}=q+N_1\frac{\partial^2 w}{\partial x^2}+N_2\frac{\partial^2 w}{\partial y^2},$$

where μ is the mass, per one unit of the panel area, q — transverse load, forces $N_1$, $N_2$, being the result of heating or some other causes,

*Here are expounded the results of research by A. A. Movchan [19].

are assumed to be constants in the entire panel and not changing with a change of sag $w(x, y, t)$.

Sag $w$ will produce over pressure $\Delta p$ on the upper streamlined surface of the panel from the side of the gas, in which the body moves, and over pressure $\Delta p'$ on the lower surface from the side of the medium, which adjoins the panel from within the body,

$$\Delta p = -\frac{\quad}{v_0}\left(V\frac{\partial w}{\partial x} - \frac{\partial w}{\partial t}\right),\quad \Delta p' = -\left(k_1 w + k_2\frac{\partial w}{\partial t}\right).$$

Here $p_0$ is the pressure, $v_0$ — speed of sound in the gas at infinity, $k_1$ and $k_2$ — non-negative numbers, characterizing properties of medium ($k_1$ — elastic support factor, $k_2$ — damping factor). The transverse load $q$ is the result of pressures shown, $q = \Delta p' - \Delta p$.

Subsequently instead of $x$, $y$, $w$ we use values $\frac{x}{a}$, $\frac{y}{b}$ and $\frac{w}{\alpha}$, for which designations $x$, $y$, $w$ are retained.

With the above assumptions for a panel supported along its entire contour we obtain the following perturbed motion equations,

$$\begin{gathered}\frac{\partial^4 w}{\partial x^4} + 2\frac{a^2}{b^2}\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{a^4}{b^4}\frac{\partial^4 w}{\partial y^4} - \frac{a^2 N_1}{D}\frac{\partial^2 w}{\partial x^2} - \frac{a^4 N_2}{b^2 D}\frac{\partial^2 w}{\partial y^2} - \\ -\frac{a^3 p_0 \varkappa V}{D v_0}\frac{\partial w}{\partial x} + \frac{a^4}{D}\left[\mu\frac{\partial^2 w}{\partial t^2} + \left(\frac{p_0\varkappa}{v_0} + k_2\right)\frac{\partial w}{\partial t} + k_1 w\right] = 0, \\ w(0, y, t) = \frac{\partial^2 w(0, y, t)}{\partial x^2} = w(1, y, t) = \frac{\partial^2 w(1, y, t)}{\partial x^2} = 0, \\ w(x, 0, t) = \frac{\partial^2 w(x, 0, t)}{\partial y^2} = w(x, 1, t) = \frac{\partial^2 w(x, 1, t)}{\partial y^2} = 0.\end{gathered} \tag{2.1}$$

For obtaining sufficient criteria of instability of undisturbed motion let us study the class of solutions

$$w(x, y, t) = X(x)\sin n\pi y e^{\omega t}, \qquad (n = 1, 2, \ldots), \tag{2.2}$$

where $\omega = p + iq$ is a complex number, $X(x) = |X(x)|e^{i\psi(x)}$ is the complex function of real value $x$. Putting (2.2) in (2.1) and introducing designations,

$$n_1 = \frac{a^2 N_1}{\pi^2 D}, \quad n_2 = \frac{a^2 N_2}{\pi^2 D}, \quad k = \frac{n^2 a^2}{b^2} + \frac{1}{2} n_1,$$
$$A = \frac{a^3 \rho_0 \varkappa V}{D a_0}, \quad B = \frac{\rho_0 \varkappa}{a_0} + k_2, \quad \lambda = -\frac{a^4}{D}(\mu\omega^2 + B\omega), \tag{2.3}$$
$$d = \pi^4 \left[\frac{1}{4} n_1^2 + \frac{n^2 a^2}{b^2}(n_1 - n_2)\right] - \frac{a^4 k_1}{D}, \quad \lambda^0 = \lambda + d,$$

we find the function (2.2) is the solution of initial problem (2.1) when and only when X(x) is the eigenfunction of the boundary-value problem,

$$X^{IV} - 2k\pi^2 X^{II} + k^2\pi^4 X - AX^I = (\lambda + d) X = \lambda^0 X,$$
$$X(0) = X^{II}(0) = X(1) = X^{II}(1) = 0, \tag{2.4}$$

and complex frequency $\omega$ is determined by formula

$$\omega = -\frac{B}{2\mu} \pm \frac{1}{\mu}\sqrt{\frac{B^2}{4} - \frac{D\mu\lambda}{a^4}}. \tag{2.5}$$

Let us note that to the complex solution (2.2) correspond real proper motions of panel with sags,

$$w(x, y, t) = |X(x)| \sin n\pi y e^{pt} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} [\psi(x) + qt]. \tag{2.2'}$$

The actual solution (2.2) is subsequently termed a complex proper motion.

Value A in equation (2.4) is called the reduced speed of undisturbed motion of the panel, $\lambda$ and $\lambda^0$ are called eigenvalues.

Complex frequencies (2.5) will be designated as $\omega$ and $\omega'$ in such a manner that $\mathrm{Re}\,\omega' \leq \mathrm{Re}\,\omega$ is fulfilled. Frequency $\omega'$ has a negative real part with any $\lambda$, and for frequency $\omega$ $\mathrm{Re}\,\omega < 0$, $\mathrm{Re}\,\omega = 0$ or $\mathrm{Re}\,\omega > 0$ is fulfilled depending upon whether $\lambda$ is inside or outside the parabola (Fig. 7),

$$\mathrm{Re}\,\lambda = \frac{D\mu}{a^4 B^2} (Im\,\lambda)^2. \tag{2.6}$$

Thus, to the eigenvalue $\lambda$ of the boundary-value problem (2.4) correspond two complex proper motions $w'(x, y, t)$ and $w(x, y, t)$, the

first of which damps with the flow of time, but the second, while damping, either has a constant amplitude or deviates without a limit depending on the fact whether $\lambda$ is inside, on, or outside the parabola of stability (2.6).

For every $k = \frac{a^2 n^2}{b^2} + 0.5n_1$, $(n = 1, 2, \ldots)$ equations (2.4) determine own boundary-value problem. Considering a great number of values of all these boundary-value problems, let us use the term of the instability of undisturbed motion of the panel for the number s of eigenvalues $\lambda$, located outside of the stability parabola. Obviously, inequality $s > 0$ means that there are proper motions of the panel, the amplitude of which grows with the passage of the time without any limit; equality $s = 0$ signifies the absence of proper motions of the panel with the growing amplitude. Let us note that we do not affirm here that when $s = 0$ the undisturbed motion is stable. If in addition to proper motions (2.2) we consider the "apparent additional motions" of form

$$[X_1(x) + tX(x)] \sin n\pi y e^{\lambda t},$$

$$\left[X_2(x) + tX_1(x) + \frac{1}{2} t^2 X(x)\right] \sin n\pi y e^{\lambda t}, \ldots,$$

which can appear for multiples of $\lambda^0$, then it can happen, that even with $s = 0$ there are deflected perturbing motions (this is possible, when the multiple $\lambda$ is on the parabola of stability).

Let us investigate eigenvalues of the boundary-value problem. The characteristic equation

$$F(k, A, \lambda^0) = 0, \tag{2.7}$$

connecting values k, and A with eigenvalues $\lambda^0$, may be, by applying

variables* α, β [18], reduced to equations;

$$F(\alpha, \beta, k) \equiv \frac{\alpha^2 \left(\operatorname{ch} 2\alpha - \operatorname{ch} \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2} \cos \beta\right)}{(\beta^2 - 3\alpha^2 + k\pi^2)^2 + 4\alpha^2\beta^2} +$$
$$+ \frac{1}{2} \frac{\left[(\beta^2 - \alpha^2 + k\pi^2)^2 + 2\alpha^2(\alpha^2 - k\pi^2)\right] \operatorname{sh} \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}}{\left[(\beta^2 - 3\alpha^2 + k\pi^2)^2 + 4\alpha^2\beta^2\right] \sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}} \frac{\sin \beta}{\beta} = 0; \tag{2.8}$$

$$A = 4\alpha(\beta^2 - \alpha^2 + k\pi^2); \tag{2.9}$$
$$\lambda^0 = k^2\pi^4 + (\alpha^2 + \beta^2)(\beta^2 - 3\alpha^2 + 2k\pi^2). \tag{2.10}$$

The characteristic system of two equations (2.8), (2.9), in which k, and A are considered to be given, and α and β are sought, has the specific property that to its every solution

$$\alpha = \alpha(k, A), \quad \beta = \beta(k, A) \tag{2.11}$$

according to the formula (2.10) corresponds the eigenvalue

$$\lambda^0 = \lambda^0(k, A), \tag{2.12}$$

i.e., the solution of equation (2.7); to every eigenvalue (2.12) correspond several solutions (2.11) of the characteristic system. With the fixed k and changing A solutions (2.11) and (2.12) can be treated as curves, which we shall term "branches." Using analytical properties of equations (2.7)-(2.10), we can show that branches (2.11) and (2.12) are continuous and "indestructible," if we consider them both in the real, and complex regions [18].

The elementary analysis of characteristic system shows that when A = 0 all eigenvalues of $\lambda^0$ are yielded by formulas

$$\lambda^0 = \pi^4 (m^2 + k)^2, \quad (m = 1, 2, \ldots) \tag{2.13}$$

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*Transition to parameters α,β may be carried out in this manner: let us assume that $z_i(k, A, \lambda^0)$ are roots of a characteristic equation. At first, we will consider certain two roots, for instance $z_1$, $z_2$, as basic parameters from parameters $z_1$, $z_2$ by means of transformation $z_1 = \alpha + i\beta$, $z_2 = \alpha - i\beta$ we pass to parameters α, β, and through them express the remaining roots and all characteristic values of the boundary-value problem.

and each eigenvalue (2.13) gives the beginning to a certain branch (2.12). Hence and from the properties of indestructibility follows the existence of a denumerable set of continuous branches

$$\lambda^0 = \lambda_m^0(k, A), \quad (m = 1, 2, \ldots), \tag{2.14}$$

which we number in such a way, that branch (2.14) with number m passes through point (2.13) with the same number m when A = 0.

Let us prove that with any fixed k and $A \neq 0$ any real eigenvalue $\lambda^0(k, A)$ is strictly larger than the least eigenvalue $\lambda^0(k, 0)$, available when A = 0. Multiplying equation (2.4) by $\tilde{X}(x)$ and integrating by parts with the use of boundary conditions, we can easily obtain the relationship

$$\lambda^0(k, A) = \frac{k^2\pi^4 I_0 + 2k\pi^2 I_1 + I_2 - iAI_3}{I_0}$$

$$\left(I_0 = \int_0^1 dx X\tilde{X}, \quad I_1 = \int_0^1 dx \frac{dX}{dx}\frac{d\tilde{X}}{dx}, \quad I_2 = \int_0^1 dx \frac{d^2X}{dx^2}\frac{d^2\tilde{X}}{dx^2}, \right.$$
$$\left. I_3 = \int_0^1 dx \frac{dX}{dx}\tilde{X}\right),$$

connecting eigenvalue $\lambda^0(k, A)$ with the corresponding eigenfunction X(x). Hence

$$\mathrm{Re}\,\lambda^0(k, A) = \frac{k^2\pi^4 I_0 + 2k\pi^2 I_1 + I_2}{I_0}, \quad |I_m \lambda^0(k, A)| = \frac{|AI_3|}{I_0}.$$

In the class of functions X(x), which are continuous together with derivatives of the fourth order and satisfy boundary conditions (2.4), the minimum $\mathrm{Re}\,\lambda^0(k, A)$ is equal to the minimum with respect to $m[\min_m \pi^4(m^2 + k)^2]$ and is attained for the solution $X(x) = \sin m\pi x$ of the boundary-value problem (2.4) when A = 0; for any solution when $A \neq 0$ the absolute inequality is fulfilled.

$$\mathrm{Re}\,\lambda^0(k, A) > \min_m \pi^4(m^2 + k)^2 = \min_m \lambda_m^0(k, 0). \tag{2.15}$$

Hence follows that which was to be proved.

The examination of the real plane $\alpha$, $\beta$ and lines, determined on it by equations (2.8)-(2.10), allows us without the fulfillment of any approximate calculations to establish the following.

For any fixed value A and for sufficiently large m all points of branches $\lambda^0 = \lambda^0_m(k, A)$ are real and positive, and with the growth of m they asymptotically come near eigenvalues (2.13), available when $A = 0$.

On the real plane A, $\lambda^0$ there exists a denumerable set of finite ovals $l_{mk}$ isolated from one another (in Fig. 9 parts of these ovals are shown in the right half-plane), consisting of real pieces of branches (2.14). In a general instance the straight line $A = 0$ intersects every oval in two certain points (2.13). For certain negative values $k \leq -2.5$ a certain oval can be pulled into a point, lying on axis $A = 0$ (for instance, the lower oval when $k = -2.5$; $-6.5$; $-12.5$; the second oval when $k = -8.5$; $-14.5$, ...). When $m^2 > -0.5k$ on each of ovals $l_{mk}$ there is a point

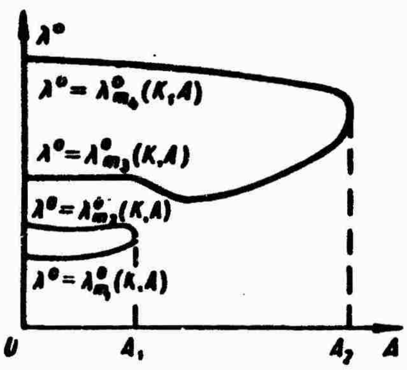


Fig. 9.

$$\lambda^{0\prime}_m(k) = \pi^4\left[(m^2 + k)^2 + \frac{1}{3}(5m^2 + k)^2\right],$$
$$A'_m(k) = \frac{8}{3\sqrt{3}}\pi^3(5m^2 + k)\sqrt{2m^2 + k}, \qquad (2.16)$$

corresponding to the solution $\alpha = \pi\sqrt{\frac{(2m^2 + k)}{3}}$, $\beta = 2m\pi$, of the characteristic system.

Let us prove the existence of complex eigenvalues $\lambda^0$ in the investigated boundary-value problem. Let us assume that $A_m = A_m(k)$ is the upper limit of those values A, with which oval $l_{mk}$ has real intersections with straight lines $A = \text{const}$. Let us consider any one

of these ovals, and for simplicity's sake let us assume that it is the lower oval and that in a certain neighborhood of the value $A = A_1$ for $A < A_1$ parts of the oval are formed by two branches,

$$\lambda^0 = \lambda^0_{m_1}(k, A), \quad \lambda^0 = \lambda^0_{m_2}(k, A). \tag{2.17}$$

When $A = A_1$ branches (2.17) cross at the point $(A_1, \lambda_1^0)$. In its neighborhood the equation (2.7) is presented in the form

$$F(k, A, \lambda^0) \equiv [(\lambda^0 - \lambda_1^0)^2 - 2\varphi_1(k, A)(\lambda^0 - \lambda_1^0) +$$
$$+ \varphi_2(k, A)]\,\Phi(k, A, \lambda^0) = 0,$$

where the analytic function $\Phi(k, A, \lambda^0)$ does not vanish in the neighborhood examined, and analytic functions $\varphi_1(k, A)$, $\varphi_2(k, A)$ turn into zero when value $A = A_1$. Hence for branches (2.17) we obtained the presentation

$$\lambda^0 = \varphi_1(k, A) \pm \sqrt{\varphi_1^2(k, A) - \varphi_2^2(k, A)} + \lambda_1^0, \tag{2.18}$$

which proves the existence of branches (2.17) in a certain neighborhood of value $A = A_1$ as well as for $A > A_1$. Inasmuch as by virtue of determination of numbers $A_m = A_m(k)$ in the neighborhood of value $A = A_1$ for $A > A_1$ branches (2.17) cannot be real, they are complex.

In those cases, when the oval is pulled in point $(A_m = 0)$, complex eigenvalues $\lambda^0$ exist with any small $A \neq 0$. For instance, with $k = -8.5$, when the second oval is pulled into a point, branches $\lambda^0 = \lambda^0_{m_3}(k, A)$, $\lambda^0 = \lambda^0_{m_4}(k, A)$ are complex with any small $A \neq 0$. This example shows, incidentally, that with a monotonous growth of A not necessarily those branches, may become complex for the first time which give the least real eigenvalues.

Considering the inequality $A_m(k) \geq A'_m(k)$, it is possible to show that for the given $k > -2$ and for any A from the interval

$$0 \leqslant A < \frac{8}{3\sqrt{3}}\pi^2(5 + k)\sqrt{2 + k} \tag{2.19}$$

all eigenvalues of $\lambda_m^0(k, A)$, m = 1, 2, ... are real.

Real parts of branches, forming ovals $l_{1k}$, $l_{2k}$, ..., and also complex parts of these branches can be built by points, finding of which by the method of successive approximations does not present any fundamental difficulties. For the series of values k from interval $-16 \leq k \leq 16$ such calculations were carried out, and for the treatment of their results presentation (2.18) was used. Here, for all k examined it was possible to select such values of constants $a_1 = a_1(k)$, $b_1 = b_1(k)$ that the expression

$$\lambda^0 = a_1\left(\frac{A^2}{A_1^2} - 1\right) \pm b_1\sqrt{1 - \frac{A^2}{A_1^2}} + \lambda_1^0, \tag{2.20}$$

obtained from (2.18), when

$$\varphi_1 = a_1\left(\frac{A^2}{A_1^2} - 1\right), \quad \varphi_2 = \varphi_1^2 + b_1\left(\frac{A^2}{A_1^2} - 1\right),$$

with an accuracy sufficient for practical calculations, approximates branches (2.17), which form the lower oval $l_{1k}$, evenly on the entire interval $0 \leq A \leq A_1$, and also in a certain interval $A_1 < A \leq A_1^*$, where $A_1^*$ is determined by conditions of accuracy. A good illustration to the above is Table 1, adduced below where k = -1 true magnitudes of eigenvalues (2.17) are given, and under them for comparison are adduced values, found with the help of (2.20)

Table 1. Magnitudes of Eigenvalues $\lambda^0$ for k = -1

| $A$ | 0 | 50 | 100 | 150 | $A$ | 190,95 | 200 | 250 | 300 | 400 | 500 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda^0$ | 877<br>877 | 867<br>867 | 834<br>834 | 760<br>760 | Re $\lambda^0$ | 519,5<br>519,5 | 528<br>527 | 578<br>577 | 714<br>711 | 800<br>794 | 1011<br>995 |
| $\lambda^0$ | 0<br>0 | 20,7<br>20,9 | 86,6<br>87,2 | 216<br>217 | Im $\lambda^0$ | 0<br>0 | 137<br>136 | 374<br>370 | 685<br>673 | 826<br>807 | 1104<br>1061 |

Number $b_1$ and combination $(\lambda_1^0 - a_1)$ included in (2.20) are found immediately;

$$b_1 = \frac{1}{2}[\lambda^0_{m_2}(k, 0) - \lambda_{m_1}(k, 0)],$$
$$\lambda_1^0 - a_1 = \frac{1}{2}[\lambda^0_{m_2}(k, 0) + \lambda_{m_1}(k, 0)], \quad (2.21)$$

inasmuch as we know $\lambda^0_{m_1}(k, 0)$, $\lambda^0_{m_2}(k, 0)$ ordinates of points of intersection of the oval $l_{1k}$ with the straight line $A = 0$. The expression (2.20) becomes definite, if in addition to (2.21) we know either the pair of numbers $A_1$, $\lambda_1^0$ (coordinates of the right most point of the oval) or $A_1$, $a_1$.

When $k = -4$ the following formulas yield value of numbers $a_1$, $A_1$ with an accuracy sufficient for practical calculations;

$$a_1 = \frac{5\pi^4}{54}(5 + 2k)^2,$$
$$A_1^2 = \frac{8\pi^6}{243}\,\frac{(5 + 2k)^3}{10 - k}[9\sqrt{8161 + 1640k + 40k^2} + 679 - 20k - 20k^2]. \quad (2.22)$$

In every concrete problem parameter $k$ passes through a sequence of values $k = \dfrac{n^2a^2}{b^2 + \frac{n_1}{2}}$, $n = 1, 2, \ldots$ Fixing $n$, we fix $k$, and also those branches, $\lambda^0 = \lambda^0_m(k, A)$, $m = 1, 2, \ldots$, which are available with this $k$. Subsequently, branches, corresponding to the fixed number $n$, will be designated $\lambda^0 = \lambda^0_{mn}(A)$.

To eigenvalues $\lambda^0_{mn}$ according to the formula (2.3) correspond eigenvalues

$$\lambda_{mn}(A) = \lambda^0_{mn}(A) - \pi^4\left[\frac{n_1^2}{4} + \frac{n^2a^2}{b^2}(n_1 - n_2)\right] + \frac{a^4k_1}{D}. \quad (2.23)$$
$$(m, n = 1, 2, \ldots)$$

The degree of instability of the unperturbed motion of the panel is equal to the number of eigenvalues (2.23) situated on the complex plane beyond the stability parabola (2.6). To every such eigenvalue

situated beyond the stability parabola, corresponds a deviating motion of the panel: the divergent motion (bulging) corresponds to the negative $\lambda$, and fluttering motion to the complex $\lambda$.

With fixed values of the parameters included in the problem only a finite number of eigenvalues [2.23] can be situated beyond the stability parabola. Beyond the stability parabola only a finite number of eigenvalues (2.23) can be situated. Actually, by virtue of the information, obtained on the asymptotic behavior of eigenvalues $\lambda^{0}_{mn}(A)$, all $\lambda_{mn}(A)$ with sufficiently large figures m or n are real, and $\lambda_{mn}(A) \to \lambda_{mn}(0)$, if at least one of the figures m or n tends toward infinity. From (2.23) by means of (2.13) we obtain:

$$\lambda_{mn}(0) = \pi^4\left[\left(m^2 + \frac{n^2a^2}{b^2}\right)^2 + m^2n_1 + \frac{n^2a^2}{b^2}n_2\right] + \frac{a^4k_1}{D}. \quad (m, n = 1, 2, \ldots)$$

Hence it is clear that for sufficiently large m or n eigenvalues $\lambda_{mn}(0)$ and $\lambda_{mn}(A)$ which approach them are positive and are located inside the stability parabola. Corresponding proper motions of the panel are oscillations with a damping amplitude.

Let us prove that in the adopted formulation of the problem, the flutter of panel exists. For proof we will examine the total parameters

$$n_1, n_2, \frac{a}{b}, \frac{a^4k_1}{D}, A, \tag{2.24}$$

which determines single-value by a great number of eigenvalues (2.23). Let us assume that this totality is such that among eigenvalues there are complex eigenvalues (as we established earlier, there always exist such A, for which the boundary-value problem (2.4) has complex eigenvalues). Without changing parameters (2.24) and, consequently, the location of eigenvalues (2.23) on the complex plane $\lambda$, let us begin to increase parameter

$$\frac{\mu D}{a^4B_1^2}=\frac{\mu D}{a^4\left(\frac{\rho_0\varkappa}{v_0}+k_2\right)^2}. \tag{2.25}$$

This is attained, for instance, either by means of increasing the mass $\mu$ or decreasing the damping factor $k_2(\sim B_1)$ of the medium, adjoining the panel from within. With the increase of parameter (2.25) branches of the stability parabola (2.6) come near to the real axis, and it is clear that, with any fixed complex (immaterial) eigenvalue, it will exceed the stability parabola with a sufficient increase of parameter (2.25) and the corresponding proper motion will be a flutter. Thus, with any forces $N_1$, $N_2$, whether compressing or stretching panel flutter is possible.

These reasonings give useful information about the influence of mass $\mu$ and damping $k_2(\sim B_1)$: loading of the panel and a decrease of damping increase the danger of flutter, lightening of the panel and an increase of damping decrease it. Let us note that lightening of the panel and an increase of damping cannot destroy either its divergent proper motions, or those flutter motions, which correspond to eigenvalues $\lambda$ with non-positive real parts.

Let us examine the effect of the elastic support factor $k_1(\sim B)$ and forces $N_1$, $N_2$. As we can see from formula (2.23), an increase $k_1$ (with other parameters unchanged) transfers all eigenvalues $\lambda_{mn}(A)$ on the complex plane to the right. Here the degree of instability changes, only in the direction of decrease if it changes at all. With a sufficient increase of $k_1$ we can render the degree of instability, equal to zero, removing the danger of all divergent and flutter motions. The same effect is produced by an increase of force $N_2$. This can be easily derived from formula (2.23), if we remember that $\lambda^0_{mn}(A)$ does not depend on $N_2$. Conversely, decrease of $N_2$ produces a

displacement of all eigenvalues $\lambda_{mn}(A)$ to the left, which increases the danger of appearance of deflecting proper motions of the panel. The large stretching force $N_1$ renders the degree of instability equal to zero. Indeed, whatever the fixed values of the remaining parameters, an increase of $N_1$ can produce such an increase of parameter k that all eigenvalues $\lambda_{mn}(A)$ will be real and close to $\lambda_{mn}(0)$, which with a sufficiently large k are all positive.

We shall prove that if with $A = 0$ the compressing forces do not exceed critical forces of buckling, then for the same compressing forces with any $A \neq 0$ divergent proper motions are impossible. We can easily obtain proof from the inequality

$$\mathrm{Re}\lambda_{mn}(A) > \min_{m}\lambda_{mn}(0), \tag{2.26}$$

which in the corollary of (2.15). Really, from the fact that with $A = 0$ compressing forces do not exceed critical forces, it follows that $\lambda_{mn}(0) \geq 0$, m, n = 1, 2, ... But then from (2.26) we obtain $\mathrm{Re}\lambda_{mn}(A) > 0$, m, n = 1, 2, ..., which is a sufficient condition of the absence of divergent to motions. Let us note that for such a panel the deflecting proper motions in flight can only be motions of the flutter type. They cannot be detected by static research, since in the static research it would be necessary to put $\lambda(A) = 0$ in equation (2.4) which is erroneous, since it contradicts the inequality $\mathrm{Re}\lambda(A) > 0$.

Inequality (2.26) also enables us to substantiate the possibility of such cases, when the panel compressed by supercritical efforts, and known to be unstable when $A = 0$ ($s > 0$ when $A = 0$), has neither the divergent nor flutter proper motions during flight with a certain speed $A \neq 0$ ($s = 0$ when $A \neq 0$). Such possibility of instances of

"stabilizing" of undisturbed motion with the growth of speed of flight will be illustrated by an example. Let us now use formula (2.16) for obtaining of certain estimates, and information, pertaining to forms of proper motions. According to (2.16) to the value of a given speed

$$A'_{mn}=\frac{8}{3\sqrt{3}}\pi^3\left(5m^2+\frac{n^2a^2}{b^2}+\frac{1}{2}n_1\right)\sqrt{2m^2+\frac{n^2a^2}{b^2}+\frac{1}{2}n_1} \tag{2.27}$$

corresponds, along with an infinite set of other solutions, an exact solution of the characteristic equation

$$\lambda'_{mn}=\pi^4\left[\left(m^2+\frac{n^2a^2}{b^2}+\frac{1}{2}n_1\right)^2+\right.$$
$$\left.+\frac{1}{3}\left(5m^2+\frac{n^2a^2}{b^2}+\frac{1}{2}n_1\right)^2\right]-d. \tag{2.28}$$

It is not difficult to find the corresponding eigenfunction,

$$X'_{mn}(x)=\sin m\pi x\sin(m\pi x+\chi)\times$$
$$\times\exp\left(-\frac{\pi}{\sqrt{3}}\sqrt{2m^2+\frac{n^2a^2}{b^2}+\frac{n_1}{2}}\,x\right),$$
$$\chi=\operatorname{arctg}\left\{3^{1/2}\left[2+\frac{1}{m^2}\left(\frac{n^2a^2}{b^2}+\frac{n_1}{2}\right)\right]^{-1/2}\right\}. \tag{2.29}$$

It is possible to show that when m = 1 and $k=\frac{n^2a^2}{b^2}+\frac{n_1}{2}\geq -1$ formulas (2.27), (2.28), (2.29) give the least real eigenvalue $\lambda'_{1n}$ and corresponding eigenfunction $X'_{1n}(x)$, which are available for the given n, and $A = A'_{1n}$. Let us compare expression $\lambda'_{1n}$, $X'_{1n}(x)$ with expressions.

$$\lambda_{1n}(0)=\pi^4\left(1+\frac{n^2a^2}{b^2}+\frac{n_1^2}{2}\right)^2-d,\quad X_{1n}(x)=\sin\pi x,$$

giving for $k \geq -1$ the least real eigenvalue $\lambda_{1n}(0)$ and corresponding eigenfunction $X_{1n}(x)$ when A = 0. We notice, first, the fact of an increase of $\lambda'_{1n}$ as compared to $\lambda'_{1n}(0)$ and, second that, whereas with A = 0 the eigenfunction $X_{1n}(x)$ does not have any zeroes in internal points of the internal $0 < x < 1$, with $A = A'_{1n}$ the eigenfunction $X'_{1n}(x)$ always turns into zero in the internal point of the same internal.

Consequently, in flight, proper motions of panels, responding to the least eigenvalues, even before these motions become flutter motions, can significantly differ both in form, and in frequency from those, which exist when the speed of undisturbed motion is zero. It is especially important to remember this, when approximation methods are applied to flutter problems. In connection with the application of approximation methods it is also useful to remember that in the presence of a sufficient compressing force with the monotonous growth of A, complex eigenvalues and flutter cannot appear for the first time in those branches $\lambda = \lambda_{mn}(A)$, which with $A = 0$ give the least eigenvalues.

If the given speeds of undisturbed motion does not exceed $A'_{1n}$, then, as it ensues from the sense of inequality (2.19), for those n, which satisfy the inequality

$$\frac{n^2a^2}{b^2} + \frac{n_1}{2} > -2,$$

all eigenvalues $\lambda_{mn}(A)$, m = 1, 2, ... are real and flutter of corresponding proper motions is impossible. Hence, taking into account (2.3), we obtain the formula of "pre-flutter" speed

$$V'_n = \frac{v_0 D}{\rho_0 \varkappa a^3} \frac{8\pi^3}{3\sqrt{3}} \left(5 + \frac{n^2a^2}{b^2} + \frac{a^2N_1}{2\pi^2 D}\right) \sqrt{2 + \frac{n^2a^2}{b^2} + \frac{a^2N_1}{2\pi^2 D}}. \qquad (2.30)$$

In a number of cases formula (2.30) enables us to clarify an essential part of the region with the zero degree of instability. For instance, if $N_2 \geq 0$ (divergent to proper motions of panel in this case are absent) for any speed V in the internal $0 \leq V \leq V'_1$, where $V'_1$ is derived from (2.30) when n = 1, flutter proper motions are impossible and the degree of instability of the undisturbed motion is equal to zero.

Formula (2.30) enables us to make a useful forewarning remark about the method of calculation of panels, greatly stretched in the direction of undisturbed motion. Departing somewhat from the adopted formulation of the problem, we will only assume here that around the rectangular panel, free from forces in its own plane, gas flows on two sides. Then, applying (2.30) with factor 0.5 before the right-hand part, we find that, no matter how great the length of a the panel in the direction of undisturbed motion, its critical flutter speed will always be larger than

$$V_1 = \frac{\pi^6}{9\sqrt{3}}\frac{a_0 E}{p_0\varkappa(1-\nu^2)}\left(\frac{h}{b}\right)^3. \tag{2.31}$$

On the other hand, if in the initial problem the panel is pre-considered to be infinitely long and at the infinitely remote end of panel we set no condition, except the condition of arbitrary smallness of initial perturbations, then we can prove for it the existence of flutter motions, when the speed exceeds the value

$$V = \pi\sqrt{\frac{E}{3(1-\nu^2)\rho}}\left(\frac{h}{b}\right), \quad \left(\rho = \frac{\mu}{h}\right). \tag{2.32}$$

The value (2.32) may be less than value (2.31) which evidences the inapplicability of formula (2.32) for limited panels. The example given shows that the results, obtained by studying panels, cylinders, etc., theoretically infinite in the direction of undisturbed motion are not always applicable to the case of finite dimensions, even if these dimensions are sufficiently great.

All the features of panel behavior in a flow which have been clarified so far, were obtained by means of qualitative research of an exact characteristic system (2.8), (2.9). In future conclusions, pertaining to branches $\lambda = \lambda_{m_1 n}(A)$, $\lambda = \lambda_{m_2 n}(A)$, which give for every fixed n the least eigenvalues, we shall use the presentation of these

branches by approximate formula (2.20), from which we obtain

$$\lambda = a_1\left(\frac{A^2}{A_1^2} - 1\right) \pm r\sqrt{a_1\left(1 - \frac{A^2}{A_1^2}\right) + \lambda_1}, \qquad (2.33)$$
$$\left(r^2 = \frac{b_1^2}{a_1^2},\ \lambda_1 = \lambda_1^0 - d\right).$$

With any A from the interval $0 \leq A \leq A_1$ eigenvalues (2.33) are real. In this interval such values of A are critical, in transition through which one of the eigenvalues (2.33) changes the sign. These values A, termed the critical divergence speeds, turn into zero the right side of (2.33) and are easily determined by the formula

$$A_{div}^2 = A_1^2\left[1 - \frac{\left(r \pm \sqrt{r^2 + 4\lambda_1}\right)^2}{4a_1}\right].$$

Hence

$$V_{div} = \frac{v_0 A_1 D}{\rho_0 a^3}\sqrt{1 - \frac{\left(r \pm \sqrt{r^2 + 4\lambda_1}\right)^2}{4a_1}}. \qquad (2.34)$$

When $A > A_1$ formula (2.33) gives complex conjugate values

$$\lambda = a_1\left(\frac{A^2}{A_1^2} - 1\right) \pm ir\sqrt{a_1\left(\frac{A^2}{A_1^2} - 1\right) + \lambda_1}, \qquad (2.35)$$

disposed on the complex plane along the second order parabolic curve

$$\operatorname{Re}\lambda = \frac{1}{r^2}(\operatorname{Im}\lambda)^2 + \lambda_1. \qquad (2.36)$$

In the interval $A_1 < A < A_1^*$, where expression (2.35) with sufficient accuracy approximates pieces of branches $\lambda = \lambda_{m_1n}(A)$, $\lambda = \lambda_{m_2n}(A)$, such value of A is critical, in transition through which eigenvalues (2.35) intersect the stability parabola (2.6). This value termed the stalling flutter speed, is derived from the condition of intersection of parabola (2.6) and (2.36):

$$A_{fl}^2 = A_1^2\left[1 + \frac{\lambda_1}{a_1(R - 1)}\right], \qquad \left(R = \frac{r^2\mu D}{a^4 B^2}\right).$$

Hence

$$V_{fl} = \frac{v_0 A_1 D}{p_0 \varkappa a^3} \sqrt{1 + \frac{\frac{\lambda_1}{a_1}}{\frac{\mu r^2 D}{a^4 B^2} - 1}} \,. \tag{2.37}$$

To every $n = 1, 2, \ldots$ corresponds a definite value of values of $A_1$, $a_1$, $\lambda_1$, $r$, depending only on the argument $k = \frac{n^2 a^2}{b^2} + \frac{n_1}{2}$ and, consequently, the specific value of stalling flutter speed (2.37), of course, if value (2.37) is real.

Let us adduce the examples which give a certain idea of the orders of stalling speeds. In all examples the following values of constants are taken $\nu = 0.3$, $\varkappa = 1.4$, $k_1 = 0$, $E = 2.1 \cdot 10^{10} \frac{kg}{mm^2}$, $p_0 = 103 \cdot 10^2 \frac{kg}{mm^2}$, $B = \frac{2p_0\varkappa}{v_0}$, $\frac{\mu}{h} = 7.8 \frac{grams}{cm^3}$, $v_0 = 340 \frac{meters}{sec}$.

<u>Example 1:</u> Square panel ($a = b$), free from forces in its own plane ($N_1 = N_2 = 0$). Results are represented graphically with solid lines (Fig. 10), depicting for $n = 1, 2, 3$ the dependency of stalling flutter speed (2.37) in m/sec on the value of the ratio $\frac{h}{a}$. The dotted line gives the value of pre-flutter speed $V_1'$, found according to formula (2.30). In regions, limited by solid curves, the degree of instability $s$ is shown. For a panel with the thickness $h = 5 \cdot 10^{-3}\,a$ we have, for instance,

$$s(0 < V < 2900) = 0, \quad s(2900 < V \leqslant 6300) = 2,$$
$$s(6300 < V < 13\,300) = 4.$$

<u>Example 2:</u> Square panel ($a = b$) with the thickness $h = 5 \cdot 10^{-3}\,a$, compressed by forces $N_1 = -\frac{4\pi^2 D}{a^2}$, $N_2 = -\frac{\pi^2 D}{a^2}$, with $V = 0$ such a panel is known to be unstable, and bulges after the least initial perturbation. Buckling becomes impossible after the achievement of stalling speed of divergence $V_{div} = 600$ m/sec, found from (2.34) when $n = 1$.

Formula (2.37) gives for n = 1, 2, 3 stalling flutter of speeds 1100, 4000, and 10,500. The degree of instability is given by realtionships,

$$s(0 < V < 600) = 1, \quad s(500 < V < 1100) = 0,$$
$$s(1100 < V < 4000) = 2, \quad s(4000 < V < 10500) = 4.$$

Comparing the above with the case of a panel, free from forces in its own plane, we see that compressing forces not only made possible the appearance of divergent proper motion, but also significantly lowered the stalling flutter speed.

The example examined is remarkable by the fact that in it the unstable state of the stationary panel, compressed by supercritical forces, is stable for the same forces in a flight at supersonic speed (from the interval 600 < V < 1100).

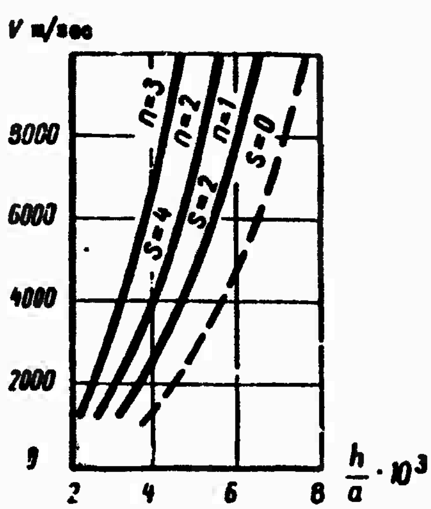


Fig. 10.

Lastly, let us note that the expounded method of investigation of the rectangular panel, supported along its entire contour is transferred without a change to those cases, when two sides of the panel, parallel to the speed of undisturbed motion, are supported, and the other two are either secured arbitrarily or are free.

In the case, when sides x = 0, x = a are fastened, characteristic equation (2.8) assumes the form

$$F(k, \alpha, \beta) \equiv \frac{ch 2\alpha - ch\sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}\cos\beta}{(\beta^2 - 3\alpha^2 + k\pi^2)^2 + 4\alpha^2\beta^2} +$$
$$+ \frac{k\pi^2 - 3\alpha^2}{(\beta^2 - 3\alpha^2 + k\pi^2)^2 + 4\alpha^2\beta^2}\,\frac{sh\sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}}{\sqrt{\beta^2 - 2\alpha^2 + 2k\pi^2}}\,\frac{\sin\beta}{\beta} = 0.$$

Adding to it relationships (2.9), (2.10), it is possible, as in the case of panel supported along its entire contour, to clarify

basic properties of branches $\lambda = \lambda_{mn}(A)$: continuity and indestructibility, asymptotic behavior ($\lambda_{mn}(A) \to \lambda_{mn}(0)$ when $m, n \to \infty$), existence of complex eigenvalues and possibility of flutter; properties of strengthening of the motion, determined by inequality (2.26), we note that the latter property in problems with other boundary conditions may not be fulfilled. Conclusions concerning the effect of parameters $k_1$, $N_1$, $k_2$, $N_2$, $\mu$ on the degree of instability remain in force. As before, to the value of a given speed (2.27) corresponds the exact solution of characteristic equation (2.28), where the corresponding eigenfunction has the form

$$X'_{mn}(x) = \sin^2 m\pi x \exp\left(-\frac{\pi}{\sqrt{3}}\sqrt{2m^2 + \frac{n^2a^2}{b^2} + \frac{\kappa_1}{2}x}\right).$$

Formulas of critical speeds (2.34), (2.37) are also retained. In Fig. 11 curves are shown, analogous to curves in Fig. 10, allowing us to judge of the degree of instability of square panels of different thicknesses, free from forces in their plane. For a panel with a thickness $h = 5 \cdot 10^{-3}$ a we have, for instance, $s(0 \leq V \leq 4600) = 0$, $s(4600 < V \leq 8100) = 2$, $s(8100 < V \leq 15200) = 4$. Comparing with the case of the panel, supported along its entire contour, we notice that in the example considered fastening of two sides resulted in a significant increase of critical flutter speeds.

Lastly, let us note that not only solutions of the examined nonselfadjoint problem, but also the solutions of corresponding selfadjoint problems can be reduced to form (2.2)'. In the latter case, as a rule, the condition $\psi(x) = \text{const}$ will be fulfilled and solutions (2.2)' will have the character of standing waves (when flutter $\psi(x) \not\equiv \not\equiv \text{const}$). An analysis of the concrete-form functions $|X(x)|$, $\psi(x)$ and

of the character of corresponding traveling waves (2.2)' during flutter is given by Movchanami (results were reported in August 1962 in the city of Stockholm and in October 1962 in the city of Yerevan). They marked a strong irregularity in the distribution of sag along the length of the panel, growth of concentration of maximum sags near the trailing edge of panel with the growth of flight speed, which agrees with the results of experiments [36]. It was determined that for panels, which have a practical significance, the speed of waves, traveling downward along the flow (it is precisely for them that the case of $p > 0$ is possible), is small as compared with the speed of sound in gas, which is in accordance with assumptions of applied aerodynamics piston theory [17].

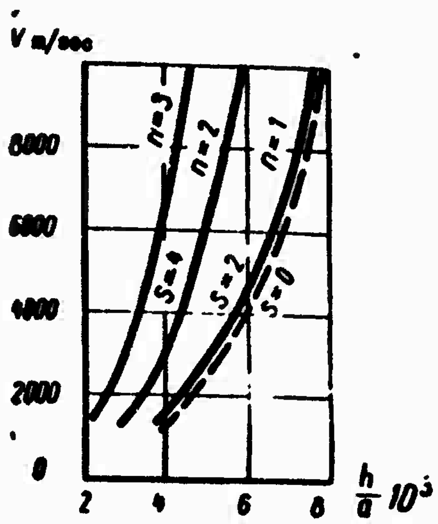


Fig. 11.

## § 3. Experimental Investigation of Panel Flutter

Experimental investigation of natural oscillations of a square flat plate in supersonic flow when values of Mach number M = 1.7; 2.3, and 3 for the case, when two edges of the plates, perpendicular to the flow are fastened, and two edges, parallel to the flow, are supported, was conducted by G. N. Mikishev [36]. Results of the experiment are in complete accord with the theoretical solution [19]. We give a description of the experiment.

Samples were prepared of steel 1Kh18N9($\sigma_b$ = 80 - 120 kg/mm$^2$) and from duralumin D16AT ($\sigma_b$ = 40 kg/mm$^2$) of 300 × 300 and 250 × 250 mm. size, of different thicknesses.

The device for bracing of samples in the wind tunnel consists of a slab, two edges of which are fixed to the walls of the pipe, the

two other edges are wedge shaped, for streamlining, the slab has a square cavity in the center. In the bottom of the cavity drain holes are made for fast levelling of pressures and for decreasing the air damping in the cavity. The sample tested is secured above the cavity. By adjusting the bracing screws of the rear cover plate and upper fulcrums it is possible to select such a position, in which edges of the plate during oscillations can converge with sufficient ease freely. The plate is blown at a zero angle of incidence. From the lower side in the attachment cavity there is motionless air. Pressure in the cavity is practically equal to the pressure in the flow. The pressure was measured in several points both in the flow and inside the cavity by mercury manometers, as well as by rheostat gauges.

For the determination of the moment of the beginning of natural oscillations, and also for the determination of the frequency and shape of oscillations resistance tensometers were used. Tensometers were glued on the lower side of the plate. Wires from tensometers were brought out through the body of the slab beyond the pipe wall.

Before every blowing frequency tests of the plate were performed by the resonance method. For this purpose the device was suspended on rubber shock absorbers. Excitation of oscillations was created by a directed mechanical vibrator, which was braced on the device. The resonance frequency was determined by the tachometer and according to the oscillogram recorded with strain gauges. The form of oscillations was determined with the help of sand. For tests in the wind tunnel only those plates were chosen, for which values of natural frequencies deviated by not more than 10% from estimated values.

The introduction of the plate into self-excited operating conditions was carried out by selecting the plate thickness and the smooth

change of pressure in the flow with the constant number M.

Observations showed that even long before the entry of the plate under intense self-excited conditions, the spectrum of natural frequencies is greatly deformed. For instance, the basic natural frequency of the plate by the moment of beginning of natural oscillations increases more than 1.5 times as compared to the frequency in motionless air. At the same time the shapes of oscillations also change. For instance, the profile of the pre-flutter shape of oscillations of the basic type in contrast to the profile in motionless air is asymmetric, and the summit of the profile is displaced toward the trailing edge. In the region of stability weak oscillations of the plate in the flow are observed. In crossing the boundary of the stability region, random oscillations are replaced by intensive natural oscillations. In natural oscillations of the plate standing waves are the form of oscillations, but under self-excited conditions plate oscillations resemble traveling waves.

Certain time plate oscillations occur with a constant amplitude. Then near the trailing edge a fatigue crack is formed, and the destruction of the plate begins. The destruction of the plate proceeds against the flow. The largest amplitudes and the fastest destruction occur for those plates, the edges of which can converge during oscillations. Limitations, set on the edge, convergence decrease the amplitudes of oscillations and sharply increase the time, necessary for the destruction of the plate.

Different methods tried for bracing of plate edges did not change the character of destruction.

Theoretical boundary of the region of stability is determined by the expression

$$\lambda = \frac{p_0 \varkappa a^3}{D} \beta_{1,2}.$$

$$\left(\beta_1 = M, \quad \beta_2 = \frac{M^2}{\sqrt{M^2 - 1}}, \quad M = \frac{V}{v_0}\right).$$

The value of parameter $\lambda$ for the basic region of stability, calculated for the square plate, 814. In Figs. 12 and 13 we give the comparison with the experiment of calculating the boundaries of the basic region of stability (the dotted curved line corresponds to the value $\beta_1$, the solid curve, to $\beta_2$). In Fig. 12 the comparison is given for the constant number M = 1.7.

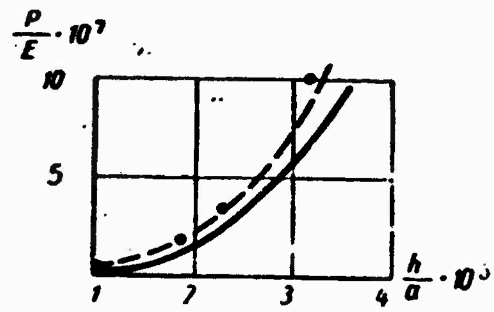



Fig. 12.

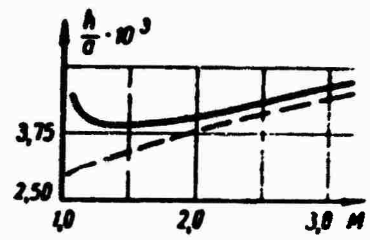



Fig. 13.

Along the X-axis the ratio between the plate thickness and its length is plotted, along the Y-axis — the ratio between the pressure and Young's modulus of the plate material. Experimental points correspond to the moment of beginning of natural oscillations. Every experimental point is obtained as the mean from several tests. The first two points correspond to steel plates, the third point — to Duralumin plates.

In Fig. 13 we show the comparison with the experiment of calculating the boundaries of the region of stability depending upon number M. Curves are plotted for Duralumin plates and pressures corresponding to the sea level.

Experimental points were also obtained by means of recalculation

for these conditions. Every experimental point corresponds to plates of such a thickness, with which natural oscillation are still produced. For thicker plates natural oscillations were not observed.

As we can see from the given comparison, the computed curves quite satisfactorily agree with the experiment.

## § 4. Unlimited Closed Cylindrical Shell

We shall seek the solution of the basic differential equation of small oscillations (1.15) for the case under consideration in the form*

$$\phi(\alpha,\beta)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}C_{kn}e^{i(n\beta+k\alpha)}, \tag{4.1}$$

where $C_{kn}$ is a certain constant number, n, k designate the number of half-waves in the meridional direction and in the direction of the generatrix of the shell respectively.

Placing (4.1) in equation (1.15), we obtain the characteristic equation, from which for $\lambda$ we obtain the following expression:

$$\lambda_1=C_0^2(k^2+n^2)^2+\frac{k^4}{(k^2+n^2)^2},\quad \lambda_2=-\frac{VBR}{Eh}k. \tag{4.2}$$

On the complex plane $\lambda_1$, $\lambda_2$ equations (4.2) depict points of parabola of the eighth degree:

$$\lambda_1=C_0^2\left[\frac{E^2h^2}{B^2R^2V^2}\lambda_2^2+n^2\right]^2+\frac{E^4h^4\lambda_2^4}{B^4R^4V^4}\left[\frac{E^2h^2}{B^2R^2V^2}\lambda_2^2+n^2\right]^{-2}. \tag{4.3}$$

For determination of the stalling speed of flow let us investigate the relative position of the parabola (4.3) with respect to the parabola of stability (1.20) in the case when n = 0 and n ≠ 0. When n = 0 (i.e., for the case, when the contour of the cross section of

*Solutions belong to R. D. Stepanov [29].

the shell remains a circle in the process of deformation) equations (4.2) take the form:

$$\lambda_1 = C_*^2 k^4 + 1, \quad \lambda_2 = -\frac{BRV}{Eh} k. \tag{4.4}$$

For points of intersection of the parabola (4.4) by the parabola of stability these qualities are true:

$$\rho \frac{R^2 q^2}{E} = C_*^2 k^4 + 1, \quad \frac{BR^2}{Eh} q = \frac{BRV}{Eh} k. \tag{4.5}$$

Excluding from the first equality (4.5) parameter q, we obtain one equation for the determination of points of mutual intersection of two investigated parabolas:

$$k^4 - \rho \frac{V^2}{EC_*^2} k^2 + \frac{1}{C_*^2} = 0, \tag{4.6}$$

the solution of which will be

$$k_{1,2,3,4} = \pm \left\{ \frac{\rho V^2}{2EC_*^2} \pm \left[ \left( \frac{\rho V^2}{2EC_*^2} \right)^2 - \frac{1}{C_*^2} \right]^{1/2} \right\}^{1/2}. \tag{4.7}$$

From (4.7) it follows that when

$$V_* > \left( \frac{2EC_*}{\rho} \right)^{1/2} \tag{4.8}$$

parabola (4.4), crossing the parabola of stability in four points, exceeds the bounds of the region of stability. Hence, when the speed of flow is larger than $\left(\frac{2\,EC_*}{\rho}\right)^{1/2}$, the shell motion may be unstable.

For the study of mutual intersection of the parabola of stability with the parabola (4.3), in a general instance when $n \neq 0$ we obtain the equation

$$k^8 + k^6 \left( 4n^2 - \rho \frac{V^2}{EC_*^2} \right) + k^4 \left( 6n^4 + \frac{1}{C_*^2} - 2\rho \frac{n^2 V^2}{EC_*^2} \right) + \\ + k^2 \left( 4n^6 - \rho \frac{n^4 V^2}{EC_*^2} \right) + n^8 = 0, \tag{4.9}$$

the solution of which will give eight roots,

$$k_1 = \pm 2n^2 \{[-a \pm (a^2 - 4b + 8n^4)^{1/2}] \pm$$
$$\pm [(-a \pm [a^2 - 4b + 8n^4]^{1/2})^2 - 16n^4]\}^{-1/2}. \tag{4.10}$$

where

$$a = 4n^2 - \rho \frac{V^2}{EC_*^2},$$
$$b = 2n^2 a + \frac{1}{C_*^2} - 2n^4. \tag{4.11}$$

Similarly to the manner in which we worked it out for the instance, when n = 0, it is possible to show here that the necessary and adequate condition, under which parabola (4.3), crossing the parabola of stability, exceeds the boundaries of the region of stability, is reduced to the determination of conditions of appearance of complex roots (4.10).

Analyzing expression (4.10), we can set the following two conditions, which are essentially different, necessary and adequate for parabola (4.3), crossing the parabola of stability, to go beyond the limits of the region of stability:

$$a^2 - 4b + 8n^4 = \rho^2 \frac{V^4}{E^2 C_*^4} - \frac{4}{C_*^2} > 0,$$
$$-a \pm (a^2 - 4b + 8n^4)^{1/2} < 4n^2. \tag{4.12}$$

For (4.12) the inequality should be fulfilled

$$a = 4n^2 - \rho \frac{V^2}{EC_*^2} > 0. \tag{4.13}$$

Inequalities (4.12) and (4.13) enable us to determine critical speeds

$$V_* > \left(\frac{2EC_*}{\rho}\right)^{1/2}, \quad n > \frac{1}{2} C_*^{-1/2} = n_*, \tag{4.14}$$
$$V_* \geq \frac{1}{2n} \left[\frac{E}{\rho} (16n^4 C_*^2 + 1)\right]^{1/2}. \tag{4.15}$$

Formula of the critical speed (4.14) identically coincides with the critical speed of the flow, found for the closed cylindrical shell when n = 0, and, as we can see from inequality (4.13) it can be used

for all values $n > n_*$, which for the class of thin shells corresponds to the number of half-waves $n > 30$ to $50$, i.e., to such a large number of half-waves, with which the shape of the cross section differs little from the circle.

The minimum of speed (4.15) with respect to $n$ occurs when $n = \frac{1}{2} C_*^{-\frac{1}{2}}$ and coincides exactly with the stalling speed, found above for $n = 0$.

Thus, the analysis performed shows that the flutter of a closed cylindrical shell of unlimited length, being in supersonic flow, can take place when the speed of flow $V > \left[\frac{2EC_*}{\rho}\right]^{1/2}$, when the shape of the cross section remains a circle.

Using formulas (1.17) and (4.2), we can obtain two values of frequencies, which essentially depend on the speed of flow,

$$\omega_{1,2} = -\frac{B}{2\rho h} \pm \left\{\left(\frac{B}{2\rho h}\right)^2 - \frac{E}{\rho R^2}\left[C_*^2 (k^2+n^2)^2 + \frac{k^4}{(k^2+n^2)^2}\right] + \right.$$
$$\left. + \frac{BVk}{\rho h} i\right\}^{1/2}. \tag{4.16}$$

The solution of differential equations for small oscillations of sloping shells (1.15), adduced in the form

$$\Phi(\alpha, \beta, t) = e^{i(n\beta + k\alpha)} e^{pt + iqt}, \tag{4.17}$$

means that along the generatrix of the shell waves propagate, running with the velocity

$$v_\phi = -\frac{q}{k}. \tag{4.18}$$

Separating the real part of the complex frequency (4.16) from the imaginary part, we find,

$$v_B = \pm \left\{\frac{1}{2k^2}\left[\pm \left\{\left[\left(\frac{B}{2\rho h}\right)^2 - \frac{E}{\rho R^2} C_*^2 (k^2+n^2)^2 + \frac{k^4}{(k^2+n^2)}\right]^2 + \right.\right.\right.$$
$$\left.\left.\left. + \frac{B^2V^2k^2}{h^2\rho^2R^2}\right\}^{1/2} - \left[\frac{B^2}{4\rho^2h^2} - \frac{E}{\rho R^2}\left(C_*^2 (k^2+n^2)^2 + \frac{k^4}{(k^2+n^2)^2}\right)\right]\right]\right\}^{1/2}. \tag{4.19}$$

Using formula (4.19), we determine the propagation velocity of the traveling wave when V = 0:

$$v_B = \pm\left\{\frac{E}{\rho R^2}\left[C_0^2\frac{(k^2+n^2)^2}{k^2} + \frac{k^2}{(k^2+n^2)^2} + \frac{B^2}{4\rho^2 h^2 R^2}\right]\right\}^{1/2}.$$

The minimum propagation velocity of the traveling wave will be when

$$n^2 = k\left[\left(\frac{1}{C_0}\right)^{1/2} - k\right]. \qquad (4.20)$$

and is equal to

$$(v_B)_{min} = \left[\frac{2EC_0}{\rho R^2} - \frac{B^2}{4\rho^2 h^2 R^2}\right]^{1/2}. \qquad (4.21)$$

Omitting all intermediate calculations, we reduce the formula of critical speed of the flow for the unlimited closed cylindrical shell, found from the examination of differential equations for small oscillations of cylindrical shells of the average length (1.11).

$$V_0 > \left[\frac{2EC_0}{\rho}\left(1 - \frac{1}{n^2}\right)\right]^{1/2}. \qquad (4.22)$$

We can use formula (4.22) for all $n \geq 2$. From (4.22) it follows that when $n = \infty$ the critical speed of the unlimited closed cylindrical shell of an average length coincides with the speed of the unlimited closed cylindrical shell, which was found by proceeding from the theory of sloping shells.

## § 5. Closed Cylindrical Shell of Limited Length

Let us investigate a series of boundary-value problems, on the basis of the differential equation of small oscillations of average-length shells (1.11).

Let us introduce a new variable $\xi$, connected with $\alpha$ by the formula

$$\alpha = \frac{l}{R}\xi, \tag{5.1}$$

where $l$ is the length of the cylindrical shell.

Then the resolving equation of small oscillations (1.11) will be written in the form

$$\frac{\partial^4\Phi_1}{\partial\xi^4} + C_0^2\frac{l^4}{R^4}\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2\frac{\partial^4\Phi_1}{\partial\beta^4} + \rho\frac{l^4}{ER^2}\frac{\partial^6\Phi_1}{\partial t^2\partial\beta^4} - \tag{5.2}$$
$$-\frac{Bl^3}{EhR^3}\left[V\frac{\partial^5\Phi_1}{\partial\xi\partial\beta^4} - l\frac{\partial^5\Phi_1}{\partial t\partial\beta^4}\right] = 0.$$

To equation (5.2) in every particular case we must adjoin boundary conditions on ends $\xi = 0$ and $\xi = 1$.

Determining by the formulas (1.6)-(1.7) displacements and internal forces of the shell through $\Phi_1$, we can present the boundary conditions for boundary-value problems in the following form:

a) the shell is supported by means of hinges on ends $\xi = 0$ and $\xi = 1$:

$$w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0, \quad M_1 = \frac{D}{R}\nu\left[\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\beta^4}\right] = 0, \tag{5.3}$$
$$(\text{when } \xi = 0 \text{ and } \xi = 1)$$

b) the shell is clamped on ends $\xi = 0$ and $\xi = 1$:

$$w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0, \quad \frac{\partial w}{\partial\xi} = \frac{R}{l}\frac{\partial^5\Phi_1}{\partial\xi\partial\beta^4} = 0; \tag{5.4}$$
$$(\text{when } \xi = 0 \text{ and } \xi = 1)$$

c) the shell (when $\xi = 0$ and $\xi = 1$) is supported by hinges on end $\xi = 1$ and is rigidly clamped on edge $\xi = 0$:

$$w=\frac{\partial^4\Phi_1}{\partial\beta^4}=0,\quad \frac{\partial w}{\partial\xi}=\frac{R}{l}\frac{\partial^5\Phi_1}{\partial\xi\,\partial\beta^4}=0,\qquad (\text{when } \xi=0)$$

$$w=\frac{\partial^4\Phi_1}{\partial\beta^4}=0,\ M_1=\frac{D}{R}\nu\left[\frac{\partial^6\Phi_1}{\partial\beta^6}+\frac{\partial^4\Phi_1}{\partial\beta^4}\right]=0;\qquad (\text{when } \xi=1) \tag{5.5}$$

d) the shell is clamped on end $\xi = 0$ and is free on end $\xi = 1$:

$$w=\frac{\partial^4\Phi_1}{\partial\beta^4}=0,\quad \frac{\partial w}{\partial\xi}=\frac{R}{l}\frac{\partial^5\Phi_1}{\partial\xi\,\partial\beta^4}=0,\qquad (\text{when } \xi=0)$$
$$T_1=\frac{EhR}{l^2}\frac{\partial^4\Phi_1}{\partial\xi^2\,\partial\beta^2}=0,\quad S=\frac{EhR^2}{l^3}\frac{\partial^4\Phi_1}{\partial\xi^3\,\partial\beta}=0\quad (\text{when } \xi=1) \tag{5.6}$$

(from the second group of equations it is clear that boundary conditions on the free edge are partially satisfied);

e) the shell is supported by means of hinges on end $\xi = 0$ and is free on edge $\xi = 1$:

$$w=\frac{\partial^4\Phi_1}{\partial\beta^4}=0,\quad M_1=\frac{D}{R}\nu\left[\frac{\partial^6\Phi_1}{\partial\beta^6}+\frac{\partial^4\Phi_1}{\partial\beta^4}\right]=0,\quad (\text{when } \xi=0)$$
$$T_1=\frac{EhR}{l^2}\frac{\partial^4\Phi_1}{\partial\xi^2\,\partial\beta^2}=0,\quad S=\frac{EhR^2}{l^3}\frac{\partial^4\Phi_1}{\partial\xi^3\,\partial\beta}=0.\ (\text{when } \xi=1). \tag{5.7}$$

In the class of solutions*

$$\Phi_1(\alpha,\beta,t)=\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}C_{kn}X_k(\xi)e^{\omega t}\cos n\beta \tag{5.8}$$

equation (5.2) after a series of simple transformations will be written in the form

$$\frac{d^4X_k}{d\xi^4}-An^4\frac{dX_k}{d\xi}+C_1^2(n^2-1)^2n^4-\lambda n^4=0, \tag{5.9}$$

where

$$-\lambda=\frac{l^4}{ER^2}\left[\rho\omega^2+\frac{B}{h}\omega\right],$$
$$A=\frac{Bl^3}{EhR^3}V,$$
$$C_1^2=C_*^2\frac{l^4}{R^4}=\frac{h^2l^4}{12R^6(1-\nu^2)}. \tag{5.10}$$

---

*The solution belongs to R. D. Stepanov [29].

The equation of the stability parabola will have the following form

$$\lambda_1 = p\frac{\mu^2 ER^6}{B^2 l^4}\lambda_2^2 \left(\lambda_1 = \frac{\mu}{ER^2} p l^2,\ \lambda_2 = -\frac{Bl^2}{EhR^2} q\right). \tag{5.11}$$

With fixed $C_1$, n, A, $\lambda$ the solution of equation (5.9), when roots of characteristic equations are different, has the following form

$$X_i(\xi) = C_1 e^{-k_1\xi} + C_2 e^{-k_2\xi} + C_3 e^{-k_3\xi} + C_4 e^{-k_4\xi}. \tag{5.12}$$

The subsequent problem is reduced to determination of nontrivial solutions of $C_i$; for this purpose it is sufficient to subordinate solution (5.12) to boundary conditions and to request conversion into zero of the corresponding determinant $\Delta(k_i)$. Dropping the question about the form of the determinant $\Delta(k_i)$ with different possible combinations of multiple roots, we will introduce into the examination a function

$$F(k_i) = \frac{\Delta(k_i)}{\delta(k_i)}, \tag{5.13}$$

where

$$\delta(k_i) = (k_1 - k_2)(k_1 - k_3)(k_1 - k_4)(k_2 - k_3)(k_2 - k_4)(k_3 - k_4).$$

From expression $\delta(k_i)$ it follows that all zeroes of function $\Delta(k_i)$ will be zeroes $\delta(k_i)$, and $F(k_i)$ will be an analytic function in the entire region of variation of variables.

The solution of equation (5.9) in the most general case is conjugate with appreciable mathematical difficulties. We will apply here the method of investigation of eigenvalues [18, 19].

The essence of the method consists of the fact that instead of solution of equation (5.9) the parameters of problem A and $\lambda$ and the two sought roots, for instance, $k_3$, $k_4$, are expressed through two

other roots $k_1$, $k_2$ of the equation,

$$
\begin{gathered}
A = -\frac{1}{n^4}\left[k^3 + k^2 k_1 + k k_1^2 + k_1^3\right], \\
\lambda = C_1^2 (n^2 - 1)^2 - \frac{k_1 k_2 k_3 k_4}{n^4}, \\
k_{3,4} = -\frac{k_1 + k_2}{2} \pm \left[k_1 k_2 - \frac{3}{4}(k_1 + k_2)^2\right]^{1/2},
\end{gathered} \tag{5.14}
$$

and instead of finding the eigenvalues of the equation (5.9) we investigate the system of two equations, of which the characteristic system is composed:

$$
\begin{gathered}
A + \frac{4\eta}{n^4}(\gamma_i^2 - \gamma^2) = 0, \\
F(\eta, \gamma) = \frac{\Delta(\eta, \gamma)}{\delta(\eta, \gamma)} = 0,
\end{gathered} \tag{5.15}
$$

where $\eta$ and $\gamma$ are values connected with roots of the equation

$$
\begin{gathered}
k_1 = \eta + i\gamma, \\
k_2 = \eta - i\gamma,
\end{gathered} \tag{5.16}
$$

besides

$$
\delta(\eta, \gamma) = 16 i \gamma \left[\gamma^2 - 2\gamma_i^2\right]^{1/2} \left[(\gamma^2 - 3\gamma_i^2) + 4\gamma_i^2 \gamma^2\right]. \tag{5.17}
$$

The left part of each equation (5.15) presents the analytic function of variables $\eta$ and $\gamma$, and the problem consists of finding such a solution

$$
\begin{gathered}
\eta_i = \eta_i(n, A), \\
\gamma_i = \gamma_i(n, A)
\end{gathered} \tag{5.18}
$$

of a system, which would enable us, using formulas:

$$
\begin{gathered}
A = -\frac{4\eta}{n^4}(\gamma_i^2 - \gamma^2), \\
k_{3,4} = -\eta \pm \left[\gamma^2 - 2\gamma_i^2\right]^{1/2}, \\
\lambda = C_1^2 (n^2 - 1)^2 + \frac{\gamma_i^2 + \gamma^2}{n^4}(\gamma^4 - 3\gamma_i^2),
\end{gathered} \tag{5.19}
$$

for every boundary-value problem to calculate corresponding eigenvalues of $\lambda$ and to establish that value of A, with which the eigenvalue becomes complex.

The easiest way to obtain a solution of the characteristic system is by the graphic method; if we plot on one drawing in a rectangular-angle system of coordinates $\eta$ and $\gamma$ graphs of the curves, determined by equations (5.15). The general appearance of curves of the characteristic system is adduced in Fig. 14; graphs of curves, corresponding to the first equation of the system (hyperbola), are plotted for different values of A = const. The subsequent problem is reduced to establishing of such values of $A_{*i}$, with which the point of the first and second real branches (5.18) coincide and we cannot draw any conclusion concerning the eigenvalues of the boundary-value problems examined.

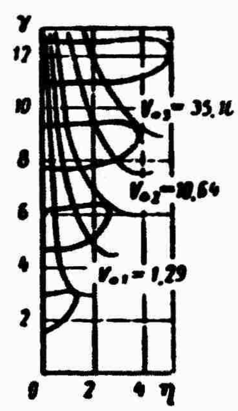



Fig. 14.

Equating $A = A_{*i}$, according to (5.10), we find the the speed of the flow, with which the stability of undisturbed motion still exists, but above which the motion can become unstable. Consequently, for every particular boundary-value problem it is necessary first of all to construct an expression of the second equation of characteristic system $\Delta(\eta, \gamma) = 0$.

Let us construct a characteristic system $\Delta(\eta, \gamma)$ in the case of the hinge supported shell. To determine non-zero $C_i$ (i = 1, 2, 3, 4) we will subordinate expression (5.12) for $X_k(\xi)$ to boundary conditions (5.3) and equate zero determinant of the system obtained:

$$\Delta(k_1, k_2, k_3, k_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ k_1^2 & k_2^2 & k_3^2 & k_4^2 \\ e^{-k_1} & e^{-k_2} & e^{-k_3} & e^{-k_4} \\ k_1^2 e^{-k_1} & k_2^2 e^{-k_2} & k_3^2 e^{-k_3} & k_4^2 e^{-k_4} \end{vmatrix} = 0.$$

Opening the determinant and performing in it the replacement of $k_1$ through $\eta$ and $\gamma$ according to formulas (5.16), we will obtain:

$$
\begin{aligned}
\text{a)}\ \Delta(\eta,\gamma) = \{&-2\eta_i^2\gamma[\gamma^2-2\eta_i^2]^{1/2}\operatorname{ch}2\eta_i+2\eta_i^2\gamma[\gamma^2-2\eta_i^2]^{1/2}\times\\
&\times\cos\gamma\operatorname{ch}[\gamma^2-2\eta_i^2]^{1/2}-(3\eta_i^4-\gamma^4-2\eta_i^2\gamma^2)\sin\gamma\, sh[\gamma^2-\\
&-2\eta_i^2]^{1/2}\}\,16i=0,
\end{aligned}
\tag{5.20}
$$

expressions $\Delta(\eta, \gamma)$ for different boundary-value problems are obtained in an analogous way,

b) the shell is clamped on ends $\xi = 0$ and $\xi = 1$:

$$
\begin{aligned}
\Delta(\eta,\gamma) = \{&\gamma[\gamma^2-2\eta_i^2]^{1/2}[\cos\gamma\operatorname{ch}(\gamma^2-2\eta_i^2)^{1/2}-\operatorname{ch}2\eta_i]+\\
&+3\eta_i^2\sin\gamma\, sh[\gamma^2-2\eta_i^2]^{1/2}\}\,8i=0;
\end{aligned}
\tag{5.21}
$$

c) the shell is clamped on end $\xi = 0$ and is hinge-supported on end $\xi = 1$:

$$
\begin{aligned}
\Delta(\eta,\gamma) = \{&2\eta_i\gamma[\gamma^2-2\eta_i^2]^{1/2}sh2\eta_i+(\gamma^2-3\eta_i^2)[\gamma^2-2\eta_i^2]^{1/2}\times\\
&\times\sin\gamma\operatorname{ch}[\gamma^2-2\eta_i^2]^{1/2}-\gamma(\gamma^2+\eta_i^2)\cos\gamma\, sh[\gamma^2-2\eta_i^2]^{1/2}\}\,8i=0;
\end{aligned}
\tag{5.22}
$$

d) the shell is clamped on end $\xi = 0$ and is free on edge $\xi = 1$:

$$
\begin{aligned}
\Delta(\eta_i,\gamma) = \{&8\gamma(\eta_i^2+\gamma^2)^2[\gamma^2-2\eta_i^2]^{1/2}ch\,2\eta_i+\\
&+4\gamma(26\eta_i^4+2\gamma^4-4\eta_i^2\gamma^2)[\gamma^2-2\eta_i^2]^{1/2}\cos\gamma\, ch[\gamma^2-2\eta_i^2]^{1/2}+\\
&+8\eta_i(2\eta_i^2\gamma^2-\gamma^4+3\eta_i^4)\sin\gamma\, sh[\gamma^2-2\eta_i^2]^{1/2}-16\eta_i\gamma(\gamma^4-\\
&-\eta_i^4)\cos\gamma\, sh[\gamma^2-2\eta_i^2]^{1/2}-16\eta_i(4\eta_i^2\gamma^2-3\eta_i^4-\gamma^4)[\gamma^2-2\eta_i^2]^{1/2}\times\\
&\times\sin\gamma\, ch[\gamma^2-2\eta_i^2]^{1/2}-32\eta_i^2\gamma^2(\gamma^2-\eta_i^2)[\gamma^2-2\eta_i^2]^{1/2}e^{-2\eta_i}\}\,i=0;
\end{aligned}
\tag{5.23}
$$

e) the shell is hinge-supported on end $\xi = 0$ and is free on end $\xi = 1$:

$$
\begin{aligned}
\Delta(\eta_i,\gamma) = \{&-2\eta_i\gamma(\gamma^2+\eta_i^2)[\gamma^2-2\eta_i^2]^{1/2}\operatorname{ch}2\gamma+\\
&+\eta_i\gamma[\gamma^2-2\eta_i^2]^{1/2}[(\gamma^2-\eta_i^2)^2+(\gamma^2-3\eta_i^2)^2]e^{-2\eta_i}+\\
&+8\eta_i^2\gamma[\gamma^2-2\eta_i^2]^{1/2}(\gamma^2-\eta_i^2)\cos\gamma\operatorname{ch}[\gamma^2-2\eta_i^2]^{1/2}+\\
&+4\eta_i(3\eta_i^2\gamma^4-\gamma^6+3\eta_i^6-5\eta_i^4\gamma^2)\sin\gamma\, sh[\gamma^2-2\eta_i^2]^{1/2}+\\
&+\gamma[5\eta_i^2\gamma^4-\gamma^6-19\eta_i^4\gamma^2+23\eta_i^6]\cos\gamma\, sh[\gamma^2-2\eta_i^2]^{1/2}+\\
&+[\gamma^2-2\eta_i^2]^{1/2}(\gamma^6+11\eta_i^4\gamma^2-\eta_i^2\gamma^4-3\eta_i^6)\sin\gamma\operatorname{ch}[\gamma^2-2\eta_i^2]^{1/2}\}\,i=0.
\end{aligned}
\tag{5.24}
$$

Let us note that when $\eta = 0$ equations $\Delta(\eta, \gamma) = 0$ degenerate

into characteristic equations of beam fundamental functions for corresponding boundary conditions.

## § 6. Effect of Aerodynamic Damping

In certain examples of calculation of panel flutter in a supersonic flow with the use of the piston theory formula [17]

$$\Delta p = \frac{p_0 \varkappa}{a_0}\left(V\frac{\partial w}{\partial x} - \frac{\partial w}{\partial t}\right)$$

aerodynamic damping $-\frac{p_0 \varkappa}{v_0} \frac{\partial w}{\partial t}$ exercises a weak effect on the value of the critical flatter velocity $V_{fl}$. This served as the cause for recommendations in favor of the quasi-stationary theory, which does not take aerodynamic damping into consideration [27,28]. However, disregard for aerodynamic damping does not allow [22] to investigate in full measure the influence of the elastic base and forces acting in the plane of the panel on the value of critical velocity and can lead to appreciable errors in its determination.

Let us show using an example of a problem on axisymmetric flutter of a circular cylindrical shell, that even in the absence of elastic support and tangential efforts, disregard for aerodynamic damping can cause incorrect results.

Let us assume that a circular cylindrical shell moves in a gas with supersonic speed along x axis directed along the axis of the cylinder (undisturbed motion), and performs additional small axisymmetric motions (perturbed motions). Applying the law of plane sections [17] in its linear formulation and the resolving equation of circular cylindrical shells [15], it is easy to obtain for dimensionless normal displacement w(x, t) of shell points the equation

$$\frac{\partial^4 w}{\partial x^4} + 2\nu\frac{a^2}{R^2}\frac{\partial^2 w}{\partial x^2} + \frac{a^4}{R^4}\left[12(1-\nu^2)\frac{R^2}{h^2} + 1\right]w - \qquad (6.1)$$

$$-\frac{a^3 p_0 \varkappa V}{D v_0}\frac{\partial w}{\partial x} + \frac{a^4}{D}\left[k_1 w + \left(k_2 + \frac{p_0 \varkappa}{v_0}\right)\frac{\partial w}{\partial t} + \mu\frac{\partial^2 w}{\partial t^2}\right] = 0.$$

Here R is the radius of the cylinder, x is a dimensionless coordinate, referred to length a of the cylinder. Let us consider natural motions — the perturbed motions of the form

$$w(x, t) = X(x)e^{\omega t}. \qquad (6.2)$$

Substituting (6.2) in (6.1) and introducing designation:

$$k = -\frac{\nu a^2}{\pi^2 R^2}, \quad A = \frac{a^3 p_0 \varkappa V}{D v_0}, \quad B = k_2 + \frac{p_0 \varkappa}{v_0},$$

$$\lambda = -\frac{a^4}{D}(B\omega + \mu\omega^2), \quad \lambda^0 = \lambda + d, \qquad (6.3)$$

$$d = -\frac{a^4 k_1}{D} - (1-\nu^2)\frac{a^4}{R^4}\left(12\frac{R^2}{h^2} + 1\right).$$

we arrive at the boundary-value problem, for the case of a cylinder freely supported (clamped) on the edges:

$$X^{IV} - 2k\pi^2 X^{II} + k^2\pi^4 X - AX^{I} = \lambda^0 X,$$

$$X(0) = X^{II}(0) = X(1) = X^{II}(1) = 0, \qquad (6.4)$$

$$(X(0) = X^{I}(0) = X(1) = X^{I}(1) = 0).$$

Comparison of equations (6.1)-(6.4) with corresponding equations (2.1)-(2.4) of this chapter shows that the problem examined about cylinder flutter is equivalent to the plane problem on the plane panel flutter of infinite amplitude, the parameters of which and condition of fastening coincide with those for a cylinder (except, of course, for the radius), while the curvature of the cylinder is compensated by an additional fictitious force, compressing the panel in its plane, and an additional fictitious elastic base. As we should expect, when $\frac{a}{R} = 0$ the identity of both problems (cylinder of infinite radius and panel of infinite amplitude) does not require the introduction of any

additional fictitious factors. If $\frac{a}{R} \neq 0$, formula (6.3), determining value k, gives with $\nu \neq 0$ a negative value, which is interpreted as a fictitious compressing force; in equality (6.3), which determines value d, an additional term appears, which is interpreted as an additional elastic base. With the decrease of radius R both fictitious factors are strengthened, which is formally expressed in a decrease (in algebraic sense) of values k and d.

As in the problem on plane panel flutter, when $B > 0$ one should distinguish two characteristic values of dimensionless speed A.

The first value $A_1(k)$ corresponds to the resonance (to the coincidence at least with respect to frequency of two different natural motions when $A < A_1(k)$) (6.2); when $A = A_1(k)$ two coiciding eigenvalues $\lambda^o$ of the boundary-value problem (6.4) become, when $A > A_1(k)$ complexly conjugate; corresponding real natural motions cease to have the shape of standing waves and take the shape of waves traveling on the shell; the amplitude of these waves damps as long as the complex eigenvalues $\lambda^o = \mathrm{Re}\lambda^o + i\mathrm{Im}\lambda^o$ are on a complex plane $\lambda^o$ inside a second degree parabola

$$\mathrm{Re}\lambda^0 = d + \frac{1}{r}(\mathrm{Im}\,\lambda^0)^2.\left(r = \frac{a^4B}{\mu D}\right). \tag{6.5}$$

The second value $A_{fl}$ corresponds to the output of complex eigenvalues of $\lambda^o$ on parabola (6.5); the amplitude of corresponding traveling waves ceases damping; it begins to increase (flutter appears), when with $A > A_{fl}$ complex eigenvalues of $\lambda^o$ exceed the limits of the rabola (6.5).

The determination of speed $A_1(k)$ usually consists of proving that with $A < A_1(k)$ all eigenvalues of $\lambda^o$ are real, but with $A > A_1(k)$ there exist complex values. Determination of the critical flutter

velocity $A_{fl}$ is appreciably more complicated, since it is necessary actually to find complex eigenvalues of $\lambda^0$ which is a very labor-consuming work.

When B = 0, i.e., in the absence of damping, both branches of parabola (6.5) merge with the real semiaxis, and consequently velocities $A_1(k)$ and $A_{fl}$ coincide. We arrive at such an essentially simpler case (as compared to case $B > 0$) usually in connection with the use in flutter calculations of quasi-stationary aerodynamic theories, which do not take into account aerodynamic damping [37, 38] into account.

Subsequently it is assumed that $k_1 = k_2 = 0$, i.e., damping of B is entirely aerodynamic, and value d is completely dependent on the curvature of the cylinder.

Obviously, the error in the appraisal of the critical flutter velocity $A_{fl}$ which appears if we disregard damping B, consists of replacing value $A_{fl}$ by a smaller value $A_1(k)$, which in no way depends on B. In the problem under consideration this error can be large owing to the following causes. As we have already said, with the decrease of the radius of cylinder R parameters k and d decrease simultaneously. The decrease of k in the interval between $2.5 \leq k \leq 0$ ($-5 \leq k \leq 0$ for clamped edges) monotonously lowers velocity $A_1(k)$ from value $A_1(0) = 343$ ($A_1(0) = 636$ for clamped edges) to zero [20]. On the other hand, the decrease of d displaces on the complex plane $\lambda^0$ the apex of the parabola (6.5) to the left; its branch in the right half-plane, where all eigenvalues $\lambda^0$ are located, move away from the real axis, which is accompanied by an increase of the least critical flutter velocity $A_{fl}$. Consequently, by the selection of radius R it is possible to lower velocity $A_1(k)$ to zero, simultaneously increasing the critical flutter velocity $A_{fl}$. Under these conditions replacement

of $A_{fl}$ by $A_1(k)$ is not permissible. The same may be said also about velocities $V_{fl}$ and $V_1$, obtained by the formula

$$V = \frac{a_0}{\rho_0 x}\frac{D}{a^3}A = \frac{a_0}{\rho_0 x}\frac{E}{12(1-\nu^2)}\left(\frac{h}{a}\right)^3 A \tag{6.6}$$

by substituting in the right-hand part values $A = A_{fl}$ and $A = A_1(k)$ respectively.

The above is illustrated by Fig. 15, in which we give the graph for the critical velocity $V_{fl}$ of axisymmetric flutter depending on the value $\frac{a}{R}$ for an aluminum cylinder clamped on the edges with the relative thickness $\frac{h}{a} = 2 \cdot 10^{-3}$. The upper curve gives the critical velocity $V_{fl}$ taking into account aerodynamic damping, caused by air at the elevator of 11-12 km above the sea level. The lower subsonic curve indicates the value of velocity $V_1$.

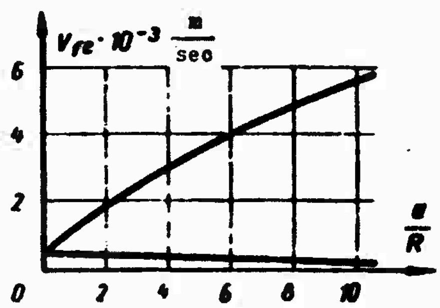


Fig. 15.

It is clear that disregarding aerodynamic damping (replacement of $V_{fl}$ by $V_1$) would lead in a number of practically interesting cases to erroneous conclusions concerning the possibility of axisymmetric flutter of the cylindrical shell during any supersonic velocities (in region of applicability of the law of plane sections).

In the plotting of graphs the results of numerical resolution of exact characteristic equations of the boundary-value problem (6.4) were used. For values of parameters of the problem, which is of interest, the first branches $\lambda_1^0(k, A)$ are located in region $0 \leq \mathrm{Re}\,\lambda^0 \leq 10^5$, $\mathrm{Im}\,\lambda^0 \geq 0$. In this region parabola (6.5), cutting off on an imaginary axis segment $\sqrt{-dr}$, is located above the straight line $\mathrm{Im}\,\lambda^0 = \sqrt{-dr} = \mathrm{const}$, which is parallel to the real axis, and differs

little from it, if the distance (-d) of its summit from the center of the coordinates is sufficiently great ($-d \geq 10^6$). In the latter case the critical flutter velocity $A_{fl}$ can be estimated from the condition of intersection of branch $\lambda_1^0(k, A)$ not with parabola (6.5), but with straight line $\mathrm{Im}\,\lambda^0 = \sqrt{-dr}$ which will lead to somewhat low results. Values (-d) and $\sqrt{-dr}$ can be conveniently calculated by approximate formulas:

$$-d = 12(1-\nu^2)\left(\frac{a}{R}\right)^2\left(\frac{a}{h}\right)^2,$$

$$\sqrt{-dr} = \frac{12(1-\nu^2)}{\sqrt{E\rho}}\,\frac{p_0\varkappa}{v_0}\,\frac{a}{R}\left(\frac{a}{h}\right)^3.$$

## § 7. Approximate Method of Investigation of Flutter. Cylindrical Panel.

Let us examine the application of the Bubnov-Galerkin method to the solution of problem on the flutter of a circular cylindrical shell of open profile, moving in a gas with supersonic velocity. It is assumed that the shell on its limiting longitudinal and lateral edges has hinged fastenings in mobile in the planes of these edges.

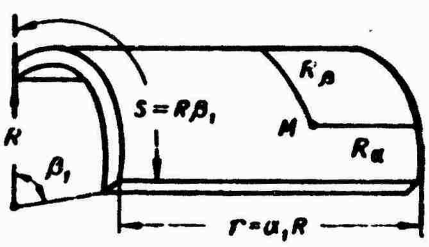


Fig. 16.

Let us assume that the shell has dimensions in the direction of generatrix $l$ and along the arc of the transverse circle s (Fig. 16). Dimensionless coordinates $\alpha$ and $\beta$ will be counted off from the point of intersection of the longitudinal edge of shell with the lateral edge. Since the shell is supported on hinges on all edges, then function $\Phi(\alpha, \beta, t)$ should be determined so that, firstly, equation be satisfied (1.10) and, secondly, on edges $\alpha = 0$, $\alpha = \alpha_1 = \frac{l}{R}$, $\beta = 0$ and $\beta = \beta_1$ the following boundary conditions be fulfilled:

$$\Phi = \frac{\partial^2\Phi}{\partial\alpha^2} = \frac{\partial^4\Phi}{\partial\alpha^4} = \frac{\partial^6\Phi}{\partial\alpha^6} \quad \text{when } \alpha = 0,\ \alpha = \alpha_1,$$
$$\Phi = \frac{\partial^2\Phi}{\partial\beta^2} = \frac{\partial^4\Phi}{\partial\beta^4} = \frac{\partial^6\Phi}{\partial\beta^6} \quad \text{when } \beta = 0,\ \beta = \beta_1. \tag{7.1}$$

Differential equation (1.10) jointly with boundary conditions (7.1) constitutes the initial boundary-value problem.

Repeating the reasoning of § 1 of this chapter we arrive at equations (1.15), (1.16) and (1.20) also, which must be examined further.

Let us apply the Bubnov-Galerkin to the solution of this boundary-value problem. Particular integrals of equation (1.15) under boundary conditions (7.1) can be determined in the following form:

$$\psi_{kn} = c_{kn} \sin\frac{k\pi\alpha}{\alpha_1} \sin\frac{n\pi\beta}{\beta_1}, \tag{7.2}$$

where $c_{kn}$ ($k = 1, 2, \ldots; n = 1, 2, \ldots$) are the coefficients sought.

Substituting (7.2) in equation (1.15), we will require that the obtained function be orthogonal to all functions $\psi_{sm}$ (when $s = 1, 2, \ldots; m = 1, 2, \ldots$).

If in equations (1.15) and (7.2) we change over to a new variable $\xi$ by the formula (5.1) and substitute (7.2) in (1.15), then after series of simple transformations we will obtain:

$$c_0^2[(a^2+n_1^2)^4 + a^4 - \lambda(a^2+n_1^2)^2] \sum_{k=1}^{\infty} c_k \sin k\pi\xi - $$
$$- Ak\pi(a^2+n_1^2)^2 \sum_{k=1}^{\infty} c_k \cos k\pi\xi = 0, \tag{7.3}$$

where

$$a = \frac{R}{l}\pi, \quad n_1 = \frac{n\pi}{\beta_1}, \quad A = \frac{BVR^3}{Ehl}. \tag{7.4}$$

In Galerkin's variational form equation (7.3) will be written in the following form:

$$\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} c_{ks}\left\{[(a^2+n_1^2)^4 + a^4 - \lambda(a^2+n_1^2)^2]\int_0^1 \sin k\pi\xi \sin s\pi\xi\, d\xi - \right.$$

$$\left. - Ak\pi(a^2+n_1^2)^2 \int_0^1 \cos k\pi\xi \sin s\pi\xi\, d\xi\right\} = 0. \qquad (7.5)$$

Integrals, included in expression (7.5), have the following values:

$$\int_0^1 \sin k\pi\xi \sin s\pi\xi\, d\xi = \begin{cases} \frac{1}{2} & \text{when } s = k \\ 0 & \text{when } s \neq k, \end{cases}$$

$$\int_0^1 \cos k\pi\xi \sin s\pi\xi\, d\xi = \begin{cases} \frac{2s}{(s^2 - k^2)\pi} & \text{when } k + s \text{ is odd} \\ 0 & \text{when } k + s \text{ is even.} \end{cases}$$

For determination of solutions of a system of linear uniform algebraic equations unequal to zero (7.5) it is necessary and sufficient to equate to zero the determinant of the system:

$$\begin{vmatrix} \frac{1}{2}(F-\lambda) & -\frac{4}{3}A & 0 & -\frac{8}{15}A & \cdots \\ \frac{4}{3}A & \frac{1}{2}(L-\lambda) & -\frac{12}{5}A & 0 & \cdots \\ 0 & \frac{12}{5}A & \frac{1}{2}(K-\lambda) & -\frac{24}{7}A & \cdots \\ \frac{8}{15}A & 0 & \frac{24}{7}A & \frac{1}{2}(M-\lambda) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 0. \qquad (7.6)$$

To every eigenvalue of equation (7.6) corresponds to the zero value $c_{ks}$ of system (7.5) and an approximate solution (7.2) of the boundary-value problem examined. In equation (7.6) we introduce the following designations:

$$F = c_0^2(a^2+n_1^2)^2 + \frac{a^4}{(a^2+n_1^2)^2}, \quad L = c_0^2(4a^2+n_1^2)^2 + \frac{2^4a^4}{(4a^2+n_1^2)^2},$$

$$K = c_0^2(9a^2+n_1^2)^2 + \frac{3^4a^4}{(9a^2+n_1^2)^2},$$

$$M = c_0^2(16a^2+n_1^2)^2 + \frac{4^4a^4}{(16a^2+n_1^2)^2}. \qquad (7.7)$$

During calculations by the first approximation, equation (7.6) assumes the form:

$$(F-\lambda)=0. \tag{7.8}$$

It follows from this that all eigenvalues of the boundary value problems examined are positive and real and, consequently, independent of the speed of flow, the undisturbed motion of the panel in class of solutions (1.14) is stable, and the critical flutter velocity of a cylindrical panel is equal to infinity.

During calculations in the second approximation from equation (7.6), being limited by determinant of the second order, we will have,

$$\lambda^2-\lambda(F+L)+FL+\frac{64}{9}A^2=0. \tag{7.9}$$

Solution of equation (7.9) will yield two roots:

$$\lambda_{12}=\frac{F+L}{2}+\left[\frac{(F-L)^2}{4}-\frac{64}{9}A^2\right]^{1/2}. \tag{7.10}$$

From formula (7.10) it follows that eigenvalues in calculations in the second approximation depend essentially on the speed of flow, and with the following values of the speed of flow

$$A=\frac{BVR^2}{Ehl}>\frac{3}{16}(F-L)$$

eigenvalues become complex, where

$$\operatorname{Re}\lambda=\lambda_1=\frac{1}{2}(F+L),\quad \operatorname{Im}\lambda=\lambda_2=\left[\frac{64}{9}A^2-\frac{(F-L)^2}{4}\right]^{1/2}. \tag{7.11}$$

Substituting values $\lambda_1$ and $\lambda_2$ in the equation of the stability parabola (1.20) and taking into account that $A = \frac{BVR^2}{Ehl}$, after a number of simple transformations we will obtain the formula for determination

of the critical velocity in the second approximation, i.e., of the speed of flow. When this speed is exceeded, there appear complex eigenvalues, lying beyond the boundaries of the stability parabola:

$$V_2 > \frac{3}{16}\frac{l}{R}\left\{\frac{E}{\rho}\left[2(F+L)+\frac{Eh^2}{B^2R^2}\,9(F-L)\right]\right\}^{1/2}. \tag{7.12}$$

In calculations in the third approximation from equation (7.6) for eigenvalues we will have

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0. \tag{7.13}$$

Here

$$\begin{aligned} I_1 &= F+L+K, \\ I_2 &= FL+FK+KL+A^2\left[\left(\frac{24}{5}\right)^2+\left(\frac{8}{3}\right)^2\right], \\ I_3 &= FKL+A^2\left[F\left(\frac{24}{5}\right)^2+K\left(\frac{8}{3}\right)^2\right]. \end{aligned} \tag{7.14}$$

Assuming that $\lambda = \lambda_1 + i\lambda_2$ and separating in equation (7.13) the real portion of complex eigenvalues from the imaginary we will obtain a system of two equations:

$$\begin{aligned} -\lambda_1^3 + 3\lambda_1\lambda_2^2 + I_1(\lambda_1^2-\lambda_2^2) - I_2\lambda_1 + I_3 &= 0, \\ -3\lambda_1^2 + \lambda_2^2 + 2I_1\lambda_1 - I_2 &= 0. \end{aligned} \tag{7.15}$$

For further computations it would be more convenient to present the equation for the stability parabola (1.20) in the form

$$r\lambda_1 = \lambda_2^2, \qquad \left(r = \frac{B^2R^2}{Eh^2\rho}\right). \tag{7.16}$$

Substituting (7.16) in equations (7.15), we will obtain the following system:

$$\begin{aligned} 3\lambda_1^2 - \lambda_1(2I_1 + r) + I_2 &= 0, \\ \lambda_1^2(I_1 + 8r) - \lambda_1(2I_2 + 3rI_1) + 3I_3 &= 0. \end{aligned} \tag{7.17}$$

For determination of the common root of two polynomials (7.17) it is necessary to equate the resultant of these equations to zero:

$$\begin{vmatrix} 3 & -(2l_1+r) & l_2 & 0 \\ 0 & 3 & -(2l_1+r) & l_2 \\ l_1+8r & -(2l_2+3rl_1) & 3l_2 & 0 \\ 0 & l_1+8r & -(2l_2+3rl_1) & 3l_2 \end{vmatrix} = 0 \tag{7.18}$$

or

$$\begin{gathered}[(l_1+8r)(2l_1+r)-3(2l_2+3rl_1)][3l_2(2l_1+r)- \\ -l_2(2l_2+3rl_1)]+[l_2(l_1+8r)-9l_2]^2=0.\end{gathered} \tag{7.19}$$

Equation (7.19) enables us to investigate the character of the change of complex eigenvalues in the boundary-value problem examined in the third approximation depending on the speed of flow and to trace their location with respect to the stability parabola.

In calculations in the fourth approximation eigenvalues are determined from an equation of the fourth degree

$$\lambda^4 - I'_1\lambda^3 + I'_2\lambda^2 - I'_3\lambda + I'_4 = 0. \tag{7.20}$$

Here

$$\begin{gathered}
I'_1 = F+L+K+M, \\
I'_2 = MF+KT+FL+MK+ML+KL+ \\
+4A^2\left[\left(\frac{12}{5}\right)^2+\left(\frac{24}{7}\right)^2+\left(\frac{8}{15}\right)^2+\left(\frac{4}{3}\right)^2\right], \\
I'_3 = FKM+MLF+MKL+KLF+4A^2\left\{\left[\left(\frac{12}{5}\right)^2+\left(\frac{24}{7}\right)^2\right]F+\right. \\
+\left[\left(\frac{12}{5}\right)^2+\left(\frac{4}{3}\right)^2\right]M+\left[\left(\frac{24}{7}\right)^2+\left(\frac{8}{15}\right)^2\right]L+ \\
\left.+\left[\left(\frac{8}{15}\right)^2+\left(\frac{4}{3}\right)^2\right]K\right\}, \\
I'_4 = FMKL+4A^2\left[\left(\frac{12}{5}\right)^2MF+\left(\frac{24}{7}\right)^2FL+\left(\frac{4}{3}\right)^2KM+\right. \\
\left.+\left(\frac{8}{15}\right)^2KL\right]+16A^4\left[\left(\frac{24}{7}\right)\left(\frac{4}{3}\right)+\left(\frac{12}{5}\right)\left(\frac{8}{15}\right)\right]^2.
\end{gathered} \tag{7.21}$$

Substituting in equation (7.20) value $\lambda = \lambda_1 + i\lambda_2$ and separating

the real portion from the imaginary, we will obtain:

$$\begin{gathered}\lambda_1^4 - I'_1\lambda_1^3 - 6\lambda_1^2\lambda_2^2 + 3I'_1\lambda_1\lambda_2^2 + \lambda_2^4 + I'_2\lambda_1^2 - I'_2\lambda_2^2 - \\ - I'_3\lambda_1 + I'_4 = 0, \\ 4\lambda_1^3 - 4\lambda_1\lambda_2^2 - 3I'_1\lambda_1^2 + I'_1\lambda_2^2 + 2I'_2\lambda_1 - I'_3 = 0.\end{gathered} \tag{7.22}$$

Replacing in equations (7.22) $\lambda_2^2$ with $r\lambda_1$ according to (7.16), we will obtain a system of equations:

$$\begin{gathered}\lambda_1^3(I'_1 + 20r) - \lambda_1^2(11rI'_1 + 4r^2 + 2I'_2) + \lambda_1(4rI'_2 + 3I'_3) - 4I'_4 = 0, \\ 4\lambda_1^3 - \lambda_1^2(3I'_1 + 4r) + \lambda_1(I'_1 r + 2I'_2) - I'_3 = 0.\end{gathered} \tag{7.23}$$

Constructing the resultant of system (7.23) and expanding it, we will obtain an equation, which will enable us to determine the critical speed of flow in the fourth approximation:

$$\begin{gathered}(a_1 0 - l_1 d)^2 + (a_1 k - c_1 l_1)^2 (c_1 0 - kd) + 2(l_1 d - a_1 0)(c_1 0 - \\ - kd)(af - bl_1) + (fd - b0)(ka_1 - c_1 l_1)(a_1 0 - l_1 d) - \\ - (fd - b0)^2 (bl_1 - a_1 f) + (c_1 0 - kd)(bk - c_1 f)(bl_1 - a_1 f) = 0.\end{gathered} \tag{7.24}$$

Here we introduce the following designations:

$$\begin{gathered}a_1 = I'_1 + 20r, \quad b = 4r^2 + 11rI'_1 + 2I'_2, \quad c_1 = 4rI'_2 + 3I'_3, \\ d = 4I'_4, \quad l_1 = 4, \quad f = 3I'_1 + 4r, \quad k = rI'_1 + 2I'_2, \quad 0 = I'_3.\end{gathered} \tag{7.25}$$

Formulas (7.12), (7.19), (7.24) enable us to calculate according to Galerkin's method (in the second, third and fourth approximations) the critical speeds of flow for cylindrical panels, supported with hinges on all edges and moving in a gas with supersonic velocity, if we know the geometric dimensions of the panel and the constant, characterizing the gas medium.

## § 8. Sloping Spherical Shell

Let us investigate natural oscillations of a sloping spherical shell in the case of support with hinges of all edges and in the case

of rigid clamping along the entire outline.*

In the class of solutions (1.14) the equation for small oscillations of a sloping spherical panel, located in a supersonic flow of gas, assumes the following form:

$$\nabla^4\Phi + \frac{1-\nu^2}{h^2R^2}\Phi + \frac{h\rho}{D}\omega^2\Phi - \frac{BV}{D}\frac{\partial\Phi}{\partial x} + \frac{B_1}{D}\omega\Phi = 0. \quad (8.1)$$

Introducing designation $\frac{h\rho}{D}\omega^2 + \frac{B_1}{D}\omega = -\lambda^{(1)}$, we will obtain the equation for the stability parabola:

$$\lambda_1 = \frac{h\rho D}{B_1^2}\lambda_2^2. \left(\lambda_1 = \frac{h\rho}{D}q^2, \quad \lambda_2 = -\frac{B_1}{D}q_1\right). \quad (8.2)$$

In a system of new dimensionless coordinates $x = a\xi$ and $y = b\eta$ equation (8.1) can be written in the following form

$$\frac{h^4}{a^4}\frac{\partial^4\Phi}{\partial\xi^4} + 2\frac{h^2}{a^2}\frac{h^2}{b^2}\frac{\partial^4\Phi}{\partial\xi^2\partial\eta^2} + \frac{h^4}{b^4}\frac{\partial^4\Phi}{\partial\eta^4} + \frac{12(1-\nu^2)h^2}{R^2}\Phi - \frac{BVh^4}{aD}\frac{\partial\Phi}{\partial\xi} - \lambda\Phi = 0. \quad (8.3)$$

where the new $-\lambda = \lambda^{(1)}h^4$, $\lambda_1 = \frac{h^5\rho}{D}q^2$, $\lambda_2 = -\frac{B_1h^4}{D}q$.

Conditions of hinged support of shell on the edges will be identically satisfied, if we look for the solution of equation (8.3) in the form

$$\Phi = \sin k\pi\xi \sin n\pi\eta. \quad (8.4)$$

Substituting (8.4) in equation (8.3), we will obtain:

$$\left[\pi\left(\frac{h^2}{a^2}k^2 + \frac{h^2}{b^2}n^2\right) - \lambda\right]\sin k\pi\xi + \frac{12(1-\nu^2)h^2}{R^2}\sin k\pi\xi - \frac{BVh^4}{aD}k\pi\cos k\pi\xi = 0. \quad (8.5)$$

Applying to equation (8.5) the Bubnov-Galerkin method for determination

*The solution of the problem is credited to R. D. Stepanov.

of eigenvalues and nonzero solutions of a system of algebraic equations, we will obtain a determinant

$$\begin{vmatrix} \frac{1}{2}(F-\lambda) & -\frac{4}{3}A & 0 & \cdots \\ \frac{4}{3}A & \frac{1}{2}(K-\lambda) & -\frac{12}{5}A & \cdots \\ 0 & \frac{12}{5}A & \frac{1}{2}(L-\lambda) & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 0,$$

where

$$F = A^4\left[\frac{h^2}{a^2}+\frac{h^2}{b^2}\right]^2 + 12(1-\nu^2)\frac{h^2}{R^2},$$

$$K = A^4\left[4\frac{h^2}{a^2}+\frac{h^2}{b^2}\right]^2 + 12(1-\nu^2)\frac{h^2}{R^2},$$

$$L = A^4\left[9\frac{h^2}{a^2}+\frac{h^2}{b^2}\right]^2 + 12(1-\nu^2)\frac{h^2}{R^2}.$$

Without reproducing here the computations, which are analogous to those which were made for the cylindrical panel, we will obtain a formula for determination of critical speeds of flow in the second approximation according to the Bubnov-Galerkin method:

$$V = \frac{3}{16}\frac{a}{h}\sqrt{\frac{E}{\rho}\frac{1}{12(1-\nu^2)}\left[2(F+K)+\frac{E\rho}{12B^2(1-\nu^2)}(F-K)^2\right]}. \tag{8.6}$$

The critical speed in the third approximation is determined from equation

$$\begin{gathered}[I_2(I_1+8r)-9I_3]^2+[(I_1+8r)(2I_1+r)-\\ -3(2I_2+3rI_1)][3I_3(2I_1+r)-I_2(2I_2+3rI_1)]=0,\end{gathered} \tag{8.7}$$

where

$$I_1 = K+F+L,$$

$$I_2 = FK+KL+FL+A^2\left[\left(\frac{24}{5}\right)^2+\left(\frac{8}{3}\right)^2\right],$$

$$I_3 = FKL+A^2\left[F\left(\frac{24}{5}\right)^2+L\left(\frac{8}{3}\right)^2\right],$$

and r is the parameter of the stability parabola: $r\lambda_1 = \lambda_2^2$.

Let us note that from the solution, obtained for the spherical panel, it is not difficult to obtain a solution for plate hinge-supported on all edges; for that purpose it is sufficient in (8.6) to make $R \to \infty$.

Now we will examine the case of natural oscillations of spherical panels rigidly clamped along their entire outline.

Here, as we did earlier, we applied the Bubnov-Galerkin method, where as approximating functions we use fundamental beam functions. It is known that beam functions, the orthogonal nature of which is well studied, do not retain this property with respect to their derivatives of the first, second and third orders, and therefore certain authors introduce the idea on quasiorthogonality of these functions, i.e., they consider integrals from the product of second derivatives of beam functions multiplied by the same function as a negligible value.* Let us note the necessity to exercise caution in postulating the property of quasiorthogonality of fundamental beam functions.

To solve this problem let us use equation (8.3) as the point of departure in which $\lambda = \lambda_1 + i\lambda_2$ are complex eigenvalues. Coordinates of the stability parabola in this case will be:

$$\lambda_1 = \frac{\rho h^6}{D} q^2, \quad \lambda_2 = -\frac{B_1 h^4}{D} q.$$

For convenience in recording let us introduce designation:

$$\frac{12(1-\nu^2)h^8}{R^8} = \frac{1-\nu^2}{c^4}\frac{h^4}{R^4}, \quad A = \frac{BVh^4}{aD}. \tag{8.8}$$

We will present the resolving function $\Phi(\xi, \eta)$ in the form of the product of beam functions

*A number of new quadratures from beam functions, encountered in the investigation of flutter in plates and shells, was calculated by R. D. Stepanov [39].

$$\Phi(\xi, \eta) = X_m(\xi) Y_n(\eta). \tag{8.9}$$

each of which satisfies both, the differential equations

$$X_m^{IV}(\xi) = \lambda_m^4 X_m(\xi), \quad Y_n^{IV}(\eta) = \lambda_n^4 Y_n(\eta),$$

and also the conditions of clamping of the spherical panel on the edges $\xi = 0$, $\xi = 1$ and $\eta = 0$, $\eta = 1$.

Substituting (8.9) in (8.3), we will obtain:

$$\begin{gathered} \frac{h^4}{a^4} \lambda_m^4 X_m Y_n + 2 \frac{h^4}{a^2 b^2} X_m'' Y_n'' + \frac{h^4}{b^4} X_m Y_n \lambda_n^4 + \\ + \left[ \frac{1-\nu^2}{c^4} \frac{h^4}{R^4} - \lambda \right] X_m Y_n - A X_m' Y_n = 0. \end{gathered} \tag{8.10}$$

If we multiply all the terms of (8.10) by $Y_n$ and integrate with respect to $\eta$ from 0 to 1, we will obtain:

$$\begin{gathered} \frac{h^4}{a^4} \lambda_m^4 X_m + 2\beta \frac{h^4}{a^2 b^2} X_m'' + \frac{h^4}{b^4} \lambda_n^4 X_m + \\ + \left[ \frac{1-\nu^2}{c^4} \frac{h^4}{R^4} - \lambda \right] X_m - A X_m' = 0, \end{gathered} \tag{8.11}$$

where

$$\beta = \frac{\int_0^1 Y_n'' Y_n \, d\eta}{\int_0^1 Y_n^2 \, d\eta}. \tag{8.12}$$

If now we multiply (8.11) by $X_s$ and integrate with respect to $\xi$ from 0 to 1, then for determination of nonzero solutions of algebraic equations

$$\begin{gathered} \sum_{m=1}^{\infty} \delta C_{ms} \left\{ \left[ \frac{h^4}{a^4} \lambda_m^4 + \frac{h^4}{b^4} \lambda_n^4 + \frac{1-\nu^2}{c^4} \frac{h^4}{R^4} - \lambda \right] X_m X_s + \right. \\ \left. + 2\beta \frac{h^4}{a^2 b^2} X_m'' X_s - A X_m' X_s \right\} = 0 \end{gathered} \tag{8.13}$$

it is necessary and sufficient to equate to zero the determinant of this system of equations, i.e.,

$$\begin{vmatrix} [F-\lambda]u_{11}+2\beta\frac{h^4}{a^2b^2}V_{11}-Aw_{11} & -Aw_{12} & 2\beta\frac{h^4}{a^2b^2}V_{13} \\ -Aw_{21} & [K-\lambda]u_{22}+2\beta\frac{h^4}{a^2b^2}V_{22}-Aw_{22} & -Aw_{23} \\ 2\beta\frac{h^4}{a^2b^2}V_{31} & -Aw_{32} & [L-\lambda]u_{33}+2\beta\frac{h^4}{a^2b^2}V_{33}-Aw_{33} \end{vmatrix}=0. \tag{8.14}$$

Here

$$\begin{aligned} F&=\lambda_1^4\frac{h^4}{a^4}+\lambda_1^4\frac{h^4}{b^4}+\frac{1-\nu^2}{c^2}\frac{h^2}{R^2},\\ K&=\lambda_2^4\frac{h^4}{a^4}+\lambda_1^4\frac{h^4}{b^4}+\frac{1-\nu^2}{c^2}\frac{h^2}{R^2},\\ L&=\lambda_3^4\frac{h^4}{a^4}+\lambda_1^4\frac{h^4}{b^4}+\frac{1-\nu^2}{c^2}\frac{h^2}{R^2}; \end{aligned} \tag{8.15}$$

$$u_{ij}=\int_0^1 X_iX_j\,d\xi,\quad v_{ij}=\int_0^1 X_i''X_j\,d\xi,\quad w_{ij}=\int_0^1 X_i'X_j\,d\xi. \tag{8.16}$$

In the case investigated these integrals have the following value:

$$\begin{aligned} &u_{11}=1{,}0359,\quad w_{12}=-3{,}399,\quad u_{22}=0{,}9984,\quad w_{23}=-5{,}512,\\ &v_{11}=-12{,}775,\ w_{13}=9{,}9065,\ v_{22}=-45{,}977,\ w_{31}=-9{,}9065,\\ &w_{11}=0,\qquad w_{21}=3{,}399,\qquad w_{22}=0,\qquad w_{32}=5{,}512. \end{aligned} \tag{8.17}$$

Every eigenvalue λ of equation (8.14) corresponds to nonzero value $C_m$ of system (8.13) and to an approximate solution of the boundary-value problem examined.

In calculations in the second approximation equation (8.14) assumes the form:

$$\begin{vmatrix} [F-\lambda]u_{11}+2\beta\frac{h^4}{a^2b^2}v_{11}-Aw_{11} & -Aw_{12} \\ -Aw_{21} & [K-\lambda]u_{22}+2\beta\frac{h^4}{a^2b^2}v_{22}-Aw_{22} \end{vmatrix}=0. \tag{8.18}$$

Expanding the determinant, we will obtain for the eigenvalues a quadratic equation

$$\lambda^2-I_1\lambda+I_2=0, \tag{8.19}$$

where

$$I_1 = F + K + 2\beta \frac{h^4}{a^2b^2}\left[\frac{v_{11}}{u_{11}} + \frac{v_{22}}{u_{22}}\right] - A\left[\frac{w_{11}}{u_{11}} + \frac{w_{22}}{u_{22}}\right].$$

$$I_2 = FK + 2\beta\frac{h^4}{a^2b^2}\left[K\frac{v_{11}}{u_{11}} + F\frac{v_{22}}{u_{22}}\right] - 2\beta A \frac{h^4}{a^2b^2}\frac{1}{u_{11}u_{22}}[w_{11}v_{22}+v_{11}w_{22}] - A\left[K\frac{w_{11}}{u_{11}}+F\frac{w_{22}}{u_{22}}\right] +$$
$$+ A^2\frac{1}{u_{11}u_{22}}[w_{11}w_{22}-w_{21}w_{12}] + 4\beta^2\frac{h^8}{a^4b^4}\frac{v_{11}v_{22}}{u_{11}u_{22}}. \tag{8.20}$$

During calculations in the third approximation for determination of eigenvalues we will obtain a cubic equation

$$\lambda^3 - \lambda^2 I'_1 + \lambda I'_2 - I'_3 = 0, \tag{8.21}$$

where

$$I'_1 = K+F+L+2\beta \frac{h^4}{a^2b^2}\left[\frac{v_{11}}{u_{11}} + \frac{v_{22}}{u_{22}} + \frac{v_{33}}{u_{33}}\right] - A\left[\frac{w_{11}}{u_{11}} + \frac{w_{22}}{u_{22}} + \frac{w_{33}}{u_{33}}\right];$$

$$\begin{aligned} I'_2 = {} & FL+KL+FK + 2\beta \frac{h^4}{a^2b^2}\left[(K+L)\frac{v_{11}}{u_{11}} + (F+L)\frac{v_{22}}{u_{22}} + \right.\\ & \left. + (F+K)\frac{v_{33}}{u_{33}}\right] - A\left[(K+L)\frac{w_{11}}{u_{11}} + (F+L)\frac{w_{22}}{u_{22}} + \right.\\ & \left. + (F+K)\frac{w_{33}}{u_{33}}\right] + 4\beta^2\frac{h^8}{a^4b^4}\left[\frac{v_{11}v_{22}}{u_{11}u_{22}} + \frac{v_{22}v_{33}}{u_{22}u_{33}} + \frac{v_{11}v_{33}}{u_{11}u_{33}} - \frac{v_{23}v_{32}}{u_{22}u_{33}}\right] - \\ & - 2\beta A\frac{h^4}{a^2b^2}\left[\frac{w_{11}v_{22}}{u_{11}u_{22}} + \frac{v_{11}w_{22}}{u_{11}u_{22}} + \frac{w_{11}v_{33}}{u_{11}u_{33}} + \frac{v_{33}w_{22}}{u_{33}u_{22}} + \frac{v_{22}w_{33}}{u_{22}u_{33}} + \frac{v_{11}w_{33}}{u_{11}u_{33}}\right] + \\ & + A^2\left[\frac{w_{11}w_{22}}{u_{11}u_{22}} + \frac{w_{11}w_{33}}{u_{11}u_{33}} + \frac{w_{22}w_{33}}{u_{22}u_{33}} - \frac{w_{21}w_{12}}{u_{11}u_{22}} - \frac{w_{23}w_{32}}{u_{22}u_{33}}\right]; \end{aligned} \tag{8.22}$$

$$\begin{aligned} I'_3 = {} & FKL - A\left[KL\frac{w_{11}}{u_{11}} + FL\frac{w_{22}}{u_{22}} + KF\frac{w_{33}}{u_{33}}\right] + \\ & + A^2\left[L\frac{w_{11}w_{22}}{u_{11}u_{22}} + K\frac{w_{11}w_{33}}{u_{11}u_{33}} + F\frac{w_{22}w_{33}}{u_{22}u_{33}} - L\frac{w_{21}w_{12}}{u_{11}u_{22}} - F\frac{w_{23}w_{32}}{u_{22}u_{33}}\right] + \\ & + 2\beta\frac{h^4}{a^2b^2}\left[FL\frac{v_{22}}{u_{22}} + KF\frac{v_{33}}{u_{33}} + KL\frac{v_{11}}{u_{11}}\right] + \\ & + 4\beta^2\frac{h^8}{a^4b^4}\left[L\frac{v_{11}v_{22}}{u_{11}u_{22}} + K\frac{v_{11}v_{33}}{u_{11}u_{33}} + F\frac{v_{22}v_{33}}{u_{22}u_{33}} - K\frac{v_{13}v_{31}}{u_{11}u_{33}}\right] - \\ & - 4\beta^2 A\frac{h^8}{a^4b^4}\left[\frac{w_{11}v_{22}v_{33}}{u_{11}u_{22}u_{33}} + \frac{v_{11}v_{33}w_{22}}{u_{11}u_{22}u_{33}} + \frac{w_{33}v_{11}v_{22}}{u_{11}u_{22}u_{33}} - \frac{w_{22}v_{31}v_{13}}{u_{11}u_{22}u_{33}}\right] + \\ & + 2\beta A^2\frac{h^4}{a^2b^2}\frac{1}{u_{11}u_{22}u_{33}}[w_{11}w_{22}v_{33} + v_{11}w_{22}w_{33} + w_{11}w_{33}v_{22} + w_{21}w_{32}v_{13} - \\ & - v_{22}w_{31}w_{13} + w_{12}w_{23}v_{31} - w_{23}w_{32}v_{11}] + \\ & + 8\beta^3\frac{h^{12}}{a^6b^6}[v_{11}v_{22}v_{33} - v_{31}v_{13}v_{22}]\frac{1}{u_{11}u_{22}u_{33}} - \\ & - A^3\frac{1}{u_{11}u_{22}u_{33}}[w_{11}w_{22}w_{33} - w_{33}w_{21}w_{12} - w_{11}w_{23}w_{32}] - \\ & - 2\beta A\frac{h^4}{a^2b^2}\left[L\frac{w_{11}v_{22}}{u_{11}u_{22}} + L\frac{v_{11}w_{22}}{u_{11}u_{22}} + F\frac{w_{22}v_{33}}{u_{22}u_{33}} + F\frac{v_{22}w_{33}}{u_{22}u_{33}} + \right.\\ & \left. + K\frac{w_{33}v_{11}}{u_{11}u_{33}} + K\frac{v_{33}w_{11}}{u_{11}u_{33}}\right]. \end{aligned}$$

The solution of equation (8.19) yields two roots

$$\lambda_{1,2} = \frac{I_1}{2} \pm \left[\frac{I_1^2}{4} - I_2\right]^{1/2}. \tag{8.23}$$

From (8.23) one can determine at what speed value do the the eigen-values of the boundary-value problem investigated become complex and the motion of the shell in a flow of gas becomes unstable:

$$\mathrm{Re}\,\lambda = \lambda_1 = \frac{I_1}{2}, \quad \mathrm{Im}\,\lambda = \lambda_2 = \left[I_2 - \frac{I_1^2}{4}\right]^{1/2}.$$

Substituting $\lambda_1$ and $\lambda_2$ in the equation of the stability parabola

$$\lambda_1 = \frac{\rho D}{B_1^2 h^3}\lambda_2^2$$

and replacing A by its value

$$A = \frac{BVh^4}{aD},$$

after a number of simple transformations we will obtain a formula for determination of the critical speed of flow in the second approximation.

The critical speed of flow in the second approximation can be obtained by another method also.

Substituting in equation (8.19) $\lambda = \lambda_1 + i\lambda_2$ and separating the real portion of the equation from the imaginary, we will obtain:

$$\begin{aligned} &\lambda_1^2 - \lambda_2^2 - I_1\lambda_1 + I_2 = 0, \\ &2\lambda_1\lambda_2 - I_1\lambda_2 = 0. \end{aligned} \tag{8.24}$$

Equation of the stability parabola is more conveniently presented in the form

$$\lambda_2^2 = \lambda_1 r,$$

where

$$r = \frac{12B^2(1-\nu^2)}{\rho E}. \tag{8.25}$$

Substituting (8.25) in (8.24), we will obtain the following system of two equations:

$$\begin{aligned} \lambda_1^2 - r\lambda_1 - I_1\lambda_1 + I_2 &= 0, \\ 2\lambda_1 - I_1 &= 0. \end{aligned} \tag{8.26}$$

For determination of a common root of two polynomials it is necessary to construct and to equate to zero the result of these equations:

$$\begin{vmatrix} 1 & -(r+I_1) & I_2 & 0 \\ 0 & -1 & -(r+I_1) & I_2 \\ 0 & 2 & -I_1 & 0 \\ 0 & 0 & 2 & -I_1 \end{vmatrix} = \begin{vmatrix} 1 & -(r+I_1) & I_2 \\ 2 & -I_1 & 0 \\ 0 & 2 & -I_1 \end{vmatrix} = 0.$$

After a number of simple transformations we will obtain an equation, from which we can determine A:

$$4I_2 - 2I_1 r - I_1^2 = 0. \tag{8.27}$$

Speed V now can be easily determined by the formula

$$V = \frac{aAD}{Bh^4}. \tag{8.28}$$

By similar means we can obtain a formula for determination of the critical speed of flow in the third approximation also:

$$\begin{aligned} &(an-ed)^3 + (ak-ce)^2(cn-kd) + 2(ed-an)(cn-kd)(af-be) + \\ &+ (fd-bn)(ka-cl)(an-ed) - (fd-bn)^2(be-af) + \\ &+ (cn-kd)(bk-cf)(be-af) = 0. \end{aligned} \tag{8.29}$$

Here

$$\begin{aligned} a &= I_1' + 20r, & e &= 4, \\ b &= 4r^2 + 11rI_1 + 2I_2', & f &= 3I_1' + 4r, \\ c &= 4rI_2' + 3I_3', & k &= rI_1' + 2I_2', \\ d &= 4I_4', & n &= I_3'. \end{aligned} \tag{8.30}$$

Formulas (8.27) and (8.29) permit us to calculate the critical speed of flow for spherical panels clamped on all their edges, in the second and third approximations, if we consider as known the geometric dimensions of the panels and the constants which characterize the gaseous medium. Let us note that if fundamental beam functions possess the property of quasiorthogonality then in all computations it is sufficient to make integrals of $v_{ij}$ equal to zero.

## § 9. Nonlinear Setting-Up and Solution of a Problem on Plate Flutter

It is of interest and at the same time extremely difficult to investigate flutter of shells in a nonlinear setting. So far we know of solutions of problems of this kind for shells. We know of the works by R. D. Stepanov [31] and B. P. Makarov [40] on the study in approximate setting of natural oscillations of a plate taking into account factors, characterizing geometric and aerodynamic nonlinearity. By analogy with determination of the critical speed of flow for problems in linear setting (see § 2 of this chapter) here we conventially consider the critical speed of flow to be such a speed, at which the envelope of perturbed solutions of a system of nonlinear differential equations of flutter in the interval of time examined constitutes a curve, which is continuously increasing in time.

For considerations of a methodical character, wishing to pay attention to the possible settings of this type of problems and methods of solving them, we adduce here a problem on the flutter of plates in a nonlinear setting [31].

Let us assume that elastic rectangular plate with sides a, b and thickness h is hinge-supported over its entire outline in such a way that the possibility of convergence or displacement of its edges is

excluded, and that a supersonic flow of gas passes about it from one side.

For the case of final sags of the plate, commensurable with its thickness h, deformations of the plate are described by the known Kármán equations:

$$\nabla^2\nabla^2\Phi = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right],$$
$$\nabla^2\nabla^2 w = \frac{h}{D}\left[\frac{\partial^2\Phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2\Phi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{q}{h}\right], \tag{9.1}$$

where w is the sag, Φ is the stress function, D is the cylindrical rigidity, and E is the elastic modulus.

For an oscillating plate, taking into account the forces of excess pressure, determined according to A. A. Il'yushin's theory [17], the normal component of the load may be written in the form

$$-q = \rho_0 h\frac{\partial^2 w}{\partial t^2} + B\frac{\partial w}{\partial t} - BV\frac{\partial w}{\partial x} - 2B_1 V\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + B_1 V^2\left(\frac{\partial w}{\partial x}\right)^2. \tag{9.2}$$

Here

$$B = \frac{\varkappa p_\infty}{V_\infty}, \quad B_1 = \frac{\varkappa(\varkappa+1)p_\infty}{4V_\infty^2},$$

$\rho_0$ is the density of material; $p_\infty$, $V_\infty$ are pressure and velocity of sound for the undisturbed gas; V is the speed of flow on the surface of the plate; $\varkappa$ is the index of the polytrope.

Equations (9.1) together with boundary conditions:

$$w = \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = u = v = 0, \quad \text{when } x = 0 \text{ и } x = a;$$
$$w = \frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = u = v = 0, \quad \text{when } y = 0 \text{ и } y = b \tag{9.3}$$

constitute the initial boundary-value problem.

Let us examine here an approximation method for the solution of

this problem, which will enable us to obtain the solution of the system in the closed form.

Upon bending, the following forces appear in the middle surface:

$$T_{xx} = \frac{Eh}{1-\nu^2}(e_{xx} + \nu e_{yy}),\quad T_{yy} = \frac{Eh}{1-\nu^2}(e_{yy} + \nu e_{xx}),$$
$$T_{xy} = \frac{Eh}{2(1+\nu)} e_{xy}, \tag{9.4}$$

where $e_{xx}$, $e_{yy}$, $e_{xy}$ are components of the ultimate deformation, determined by formulas:

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,$$
$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \tag{9.5}$$
$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

If we substitute (9.4), taking into account (9.5), in equations of motion of a two-dimensional problem

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} - \rho_0 h \frac{\partial^2 u}{\partial t^2} = 0,$$
$$\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} - \rho_0 h \frac{\partial^2 v}{\partial t^2} = 0 \tag{9.6}$$

and to study the form of the bend of a plate during oscillations in the form

$$w(x, y, t) = f(t)\psi(x, y), \tag{9.7}$$

then we can write out equation of the bend of a plate for the case of ultimate sags (9.1) in the form of a system of equations, connecting u, v and w.

$$\frac{\partial^4 w}{\partial \xi^4} + 2k^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} k^4 = \frac{k^2 h}{D}\left[\frac{\partial^2 \Phi}{\partial \eta^2}\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \xi^2}\frac{\partial^2 w}{\partial \eta^2} - 2\frac{\partial^2 \Phi}{\partial \xi \partial \eta}\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{q}{k^2 h}\right],$$
$$\frac{\partial^2 u}{\partial \xi^2} + \frac{k^2(1-\nu)}{2}\frac{\partial^2 u}{\partial \eta^2} + \frac{k(1+\nu)}{2}\frac{\partial^2 v}{\partial \xi \partial \eta} - \frac{a^2\rho_0(1-\nu^2)}{E}\frac{\partial^2 u}{\partial t^2} + E_\xi = 0, \tag{9.8}$$
$$k^2\frac{\partial^2 v}{\partial \eta^2} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial \xi^2} + \frac{k(1+\nu)}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{a^2\rho_0(1-\nu^2)}{E}\frac{\partial^2 v}{\partial t^2} + F_\eta = 0.$$

The equations are written in variables $\xi$ and $\eta$, which are connected with the old variables x and y by relationships,

$$\xi=\frac{x}{a},\quad \eta=\frac{y}{b},\quad k=\frac{a}{b}.$$

In system (9.8) we introduce designations

$$F_{\xi}=\frac{f^{2}(t)}{a}\left[\frac{\partial\psi}{\partial\xi}\left(\frac{\partial^{2}\psi}{\partial\xi^{2}}+\frac{k^{2}(1-\nu)}{2}\frac{\partial^{2}\psi}{\partial\eta^{2}}\right)+\frac{k^{2}(1+\nu)}{2}\frac{\partial^{2}\psi}{\partial\xi\partial\eta}\frac{\partial\psi}{\partial\eta}\right],$$
$$F_{\eta}=\frac{f^{2}(t)}{b}\left[\frac{\partial\psi}{\partial\eta}\left(k^{2}\frac{\partial^{2}\psi}{\partial\eta^{2}}+\frac{(1-\nu)}{2}\frac{\partial^{2}\psi}{\partial\xi^{2}}\right)+\frac{(1+\nu)}{2}\frac{\partial^{2}\psi}{\partial\xi\partial\eta}\frac{\partial\psi}{\partial\xi}\right], \tag{9.9}$$

where $\psi(\xi, \eta)$ is a function, selected in such a manner that we would know that boundary conditions (9.3) are satisfied, and f(t) is an unknown time function.

If in system (9.8) we drop the terms which take into account the longitudinal forces of inertia, and represent function w in the form of a series

$$w(\xi,\eta)=\sum_{m}\sum_{n}f_{mn}\sin m\pi\xi\sin n\pi y,$$

then the solution of the two last equations of system (9.8) will have the form:

$$u(\xi,\eta,t)=\frac{f^{2}m\pi}{16a}\sin 2m\pi\xi\left[\cos 2n\pi\eta-1+\frac{\nu n^{2}k^{2}}{m^{2}}\right]+u_{0},$$
$$v(\xi,\eta,t)=\frac{f^{2}n\pi}{16b}\sin 2n\pi\eta\left[\cos 2m\pi\xi-1+\frac{\nu m^{2}}{n^{2}k^{2}}\right]+v_{0}, \tag{9.10}$$

where $u_0$, and $v_0$ are the solution of a uniform system. For axial forces in accordance with (9.4) and (9.5), with (9.10) we will obtain:

$$T_{xx}=\frac{Eh}{1-\nu^{2}}\frac{m^{2}\pi^{2}}{8a^{2}}f_{mn}^{2}\left[1+\frac{\nu k^{2}n^{2}}{m^{2}}+(\nu^{2}-1)\cos 2n\pi\eta\right]+T_{xx}^{0},$$
$$T_{yy}=\frac{Eh}{1-\nu^{2}}\frac{n^{2}\pi^{2}}{8b^{2}}f_{mn}^{2}\left[1+\frac{\nu^{2}m^{2}}{n^{2}k^{2}}-(1-\nu^{2})\cos 2m\pi\xi\right]+T_{yy}^{0}, \tag{9.11}$$
$$T_{xy}=\frac{Eh}{2(1+\nu)}\left[\frac{\partial u_{0}}{b\partial\eta}+\nu\frac{\partial v_{0}}{a\partial\xi}\right]=T_{xy}^{0}.$$

Here

$$T^0_{xx} = \frac{Eh}{1-\nu^2}\left[\frac{\partial u_0}{a\partial\xi} + \nu\frac{\partial v_0}{b\partial\eta}\right].$$

$$T^0_{yy} = \frac{Eh}{1-\nu^2}\left[\frac{\partial v_0}{b\partial\eta} + \nu\frac{\partial u_0}{a\partial\xi}\right].$$

Taking into account Airy's relationship $T_{xx} = h\,\frac{\partial^2\Phi}{\partial\eta^2}$, $T_{yy} = h\,\frac{\partial^2\Phi}{\partial\xi^2}$, $T_{xy} = -h\,\frac{\partial^2\Phi}{\partial\xi\partial\eta}$ and expression (9.2) for a normal component of load, we will represent the first equation of equilibrium of the Kármán system of (9.1) in the form

$$\begin{aligned}
&\frac{\partial^4 w}{\partial\xi^4} + 2k^2\frac{\partial^4 w}{\partial\xi^2\partial\eta^2} + k^4\frac{\partial^4 w}{\partial\eta^4} = \frac{Eh\pi^2}{8D(1-\nu^2)}\Big\{\sum_m\sum_n f^2_{mn}m^2\Big[1 + \\
&+ \frac{\nu k^2 n^2}{m^2} + (\nu^2-1)\cos 2n\pi\eta\Big]\frac{\partial^2 w}{\partial\xi^2} + \sum_m\sum_n k^4 n^2 f^2_{mn}\Big[1 + \frac{\nu^2 m^2}{n^2k^2} + \\
&+ (\nu^2-1)\cos 2m\pi\xi\Big]\frac{\partial^2 w}{\partial\eta^2}\Big\} - \frac{\rho_0 a^4 h}{D}\frac{\partial^2 w}{\partial t^2} - \frac{a^4}{D}B\frac{\partial w}{\partial t} + \\
&+ \frac{BVa^3}{D}\frac{\partial w}{\partial\xi} + \frac{2B_1Va^3}{D}\frac{\partial w}{\partial t}\frac{\partial w}{\partial\xi} - \frac{B_1V^2a^2}{D}\left(\frac{\partial w}{\partial\xi}\right)^2.
\end{aligned} \tag{9.12}$$

Equation (9.12) is correct for plate which is hinge-supported all over its outline on immovable supports, within the limits of the approximate solution proposed. The degree of approximation of the solution obtained consists of the fact that, everywhere the solution of the uniform system $u_0$ and $v_0$ is assumed to be equal to zero while the boundary conditions of the two-dimensional problem are satisfied not continuously, but at separate points of the outline of the plate. Actually, from expressions (9.10) with the above assumption it follows that if on edges $\eta = 0$ and $\eta = 1$ ($\xi = 0$ and $\xi = 1$) $v(u)$ is identically equal to zero, then component of displacement $u(v)$ turns into zero only in separate points of the outline of the plate, although the total displacement $u(v)$ on the corresponding edge is equal to zero.

For the case of a two-term approximation of function w with respect to variable $\xi$ and its monomial approximation with respect to $\eta$,

i.e., for the case of a cylindrical bend of plate with respect to variable $\eta$, applying to the equation (9.12) the Bubnov-Galerkin method, we will obtain the following system of nonlinear second-order differential equations, describing the phenomenon of plate flutter

$$
\begin{gathered}
\ddot{\varphi}_1 + M\dot{\varphi}_1 - \frac{8}{3}M\frac{V}{V_0}\varphi_2 + \Omega(1+k^2)^2\varphi_1 - \frac{8}{3}M\frac{V}{V_0}\dot{\varphi}_1\varphi_2 + \\
+ \frac{32}{9}M_1\frac{V^2}{V_0^2}\varphi_1^2 + \frac{2}{3}\Omega(\chi_1 + k^4\chi_2)\varphi_1 = 0, \\
\ddot{\varphi}_2 + M\dot{\varphi}_2 + \frac{8}{3}M\frac{V}{V_0}\varphi_1 + \Omega(4+k^2)^2\varphi_2 + \frac{896}{45}M_1\frac{V^2}{V_0^2}\varphi_2\varphi_1 - \\
- \frac{2}{3}\Omega(k^4\chi_2 + 4\chi_1)\varphi_2 + \frac{4}{3}\Omega k^4(s_2\chi_2 - s_1\chi_1)\varphi_2 = 0.
\end{gathered}
\tag{9.13}
$$

Equations (9.13) are written in dimensionless variables $\xi$, $\eta$, $\tau = \frac{V_0 t}{a}$, $\varphi_1 = \frac{f_1}{h}$, $\varphi_2 = \frac{f_2}{h}$. Here we introduce designations:

$$
\begin{gathered}
M = \frac{Ba}{\rho_0 h V_0}, \quad M_1 = \frac{B_1}{\rho_0}, \quad \Omega = \frac{E}{12\rho_0(1-\nu^2)}\frac{\pi^4}{V_0}\frac{h^3}{a^3}, \\
\chi_1 = \varphi_1^2\left(\frac{3}{2} - \frac{\nu^2}{2} + \nu k^2\right) + 4\varphi_2^2\left(\frac{3}{2} + \frac{\nu k^2}{4} - \frac{\nu^2}{2}\right), \\
\chi_2 = \varphi_1^2\left(\frac{3}{2} - \frac{\nu^2}{2} + \frac{\nu}{k^2}\right) + \varphi_2^2\left(1 + \frac{4\nu}{k^2}\right), \quad k = \frac{a}{b}, \\
s_1 = \frac{\frac{5}{2} + \frac{5\nu}{k^2} - \frac{\nu^2}{2}}{\Delta}, \quad s_2 = \frac{\frac{15}{2} + 2\nu k^2 - \frac{5\nu^2}{2}}{\Delta}, \\
\Delta = (\nu^2 - 1)\left(\frac{15}{2} - \nu^2 + \frac{\nu k^2}{2}\right).
\end{gathered}
\tag{9.14}
$$

Further the system (9.13) by replacement of variables $\frac{d\varphi_i}{d\tau} = u_i$ $(i = 1, 2)$ is reduced to a system of four nonlinear first-order differential equations, the integration of which can be carried out on a computer with specific initial conditions, which was done with the precision of $10^{-4}$ at the interval of dimensionless time $0 \leq \tau \leq 40$.

For the above accuracy the magnitude of the step of integration did not exceed 0.2. Automatic selection of step in the Runge-Kutta method was produced in the following manner. In the initial step h

was calculated by solution of system $\tilde{\varphi}_i$ at point $\xi_0 + h$, then the step was divided in half and $\tilde{\varphi}_i$ was calculated at point $\xi_0 + \frac{h}{2}$. By value $\tilde{\varphi}_i$ and step $\frac{h}{2}$ we found a new solution $\tilde{\tilde{\varphi}}_i$ at point $\xi_0 + h$ once again. The accuracy of the solution was checked by two values $\tilde{\varphi}_i$ and $\tilde{\tilde{\varphi}}_i$, calculated at one point $\xi_0 + h$. If difference of solutions does not exceed the prescribed accuracy, then a recalculation of the solution is performed with a half step, if however the required accuracy is attained, then we check whether it is possible to perform the further computation with a doubled step of whether the step should remain the same. Solution of the system of differential equations by this method was conducted for a plate, having the following relative dimensions:

$$k = \frac{a}{b} = 3, \quad \frac{h}{a} = \frac{1}{400};$$

with constants of problem:

$$\varkappa = 1{,}4; \quad E = 2\cdot 10^6\ \frac{\text{kg}}{\text{cm}^2}; \quad \rho_0 = 7{,}8\cdot 10^{-3}\ \frac{\text{kg}}{\text{cm}^3};$$
$$p_\infty = 1{,}014\ \frac{\text{kg}}{\text{cm}^2}; \quad V_0 = V_\infty = 3{,}4\cdot 10^4\ \frac{\text{cm}}{\text{sec}}$$

and with initial conditions:

$$1.\ \dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0:$$
$$\text{a)} \begin{cases} \varphi_1(0) = 0{,}1 \\ \varphi_2(0) = 0, \end{cases} \quad \text{б)} \begin{cases} \varphi_1(0) = 0{,}1 \\ \varphi_2(0) = 0{,}1, \end{cases} \quad \text{b)} \begin{cases} \varphi_1(0) = 1 \\ \varphi_2(0) = -1; \end{cases} \tag{9.15}$$

$$2.\ \varphi_1(0) = \varphi_2(0) = \dot{\varphi}_2(0) = 0:$$
$$\text{a)}\ \dot{\varphi}_1(0) = 0{,}04, \qquad \text{б)}\ \dot{\varphi}_1(0) = 0{,}4. \tag{9.16}$$

The problem consisted of the fact that, with specific initial conditions of the boundary-value problem examined, we were to find the perturbed solution of system (9.13) for various values of speeds of flow and to establish the speed, at which solutions continuously increasing in time first appear in the time interval under consideration.

Results of calculations show that other conditions being equal, the value of the critical speed for a plate in nonlinear setting is essentially dependent on initial conditions.

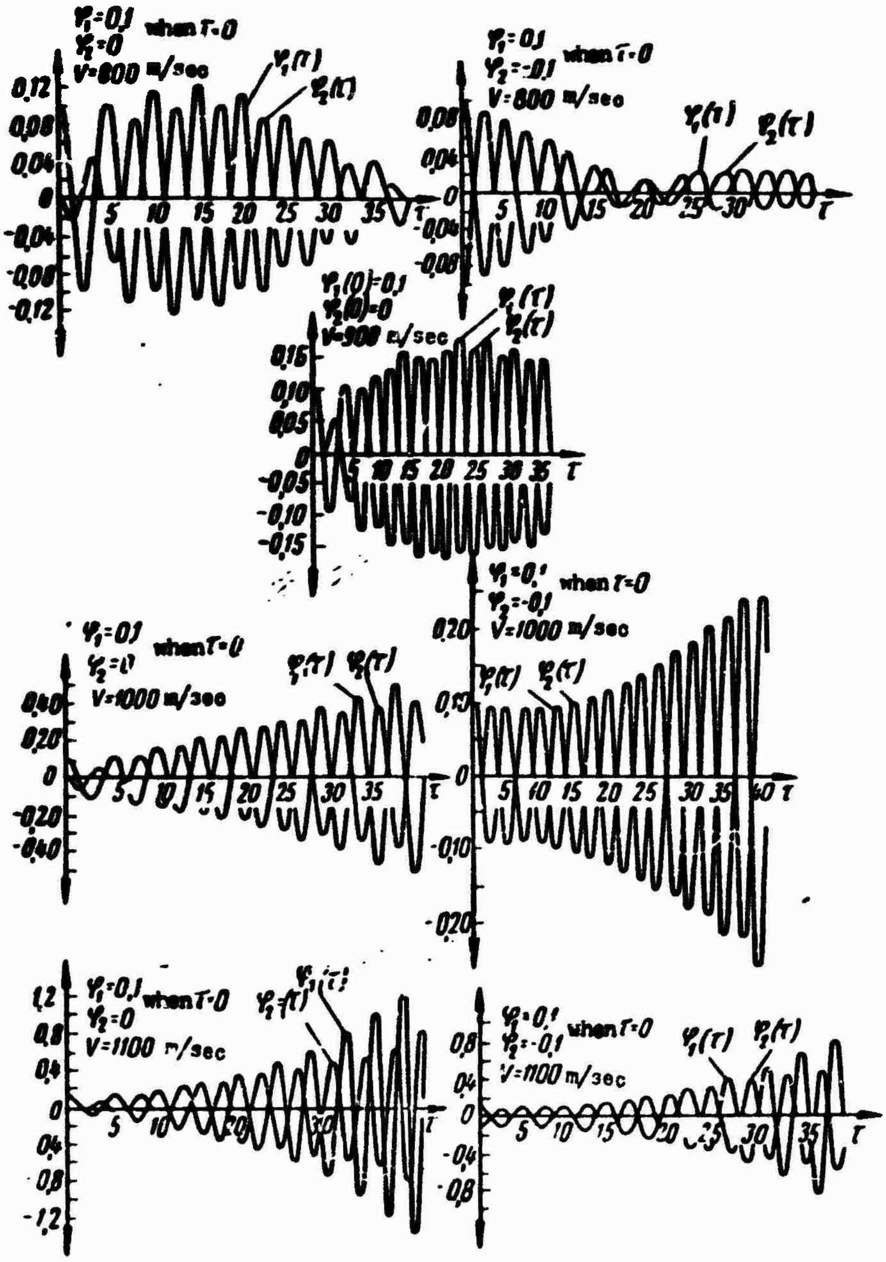



Fig. 17.

In Figs. 17 and 18 we adduce graphs of perturbed solutions of the initial system of differential equations (9.13) for speeds of flow of 800, 1000, 1200, 1400 and 1600 m/sec and under initial conditions, prescribed in the form of the initial sag (9.15), changing with respect

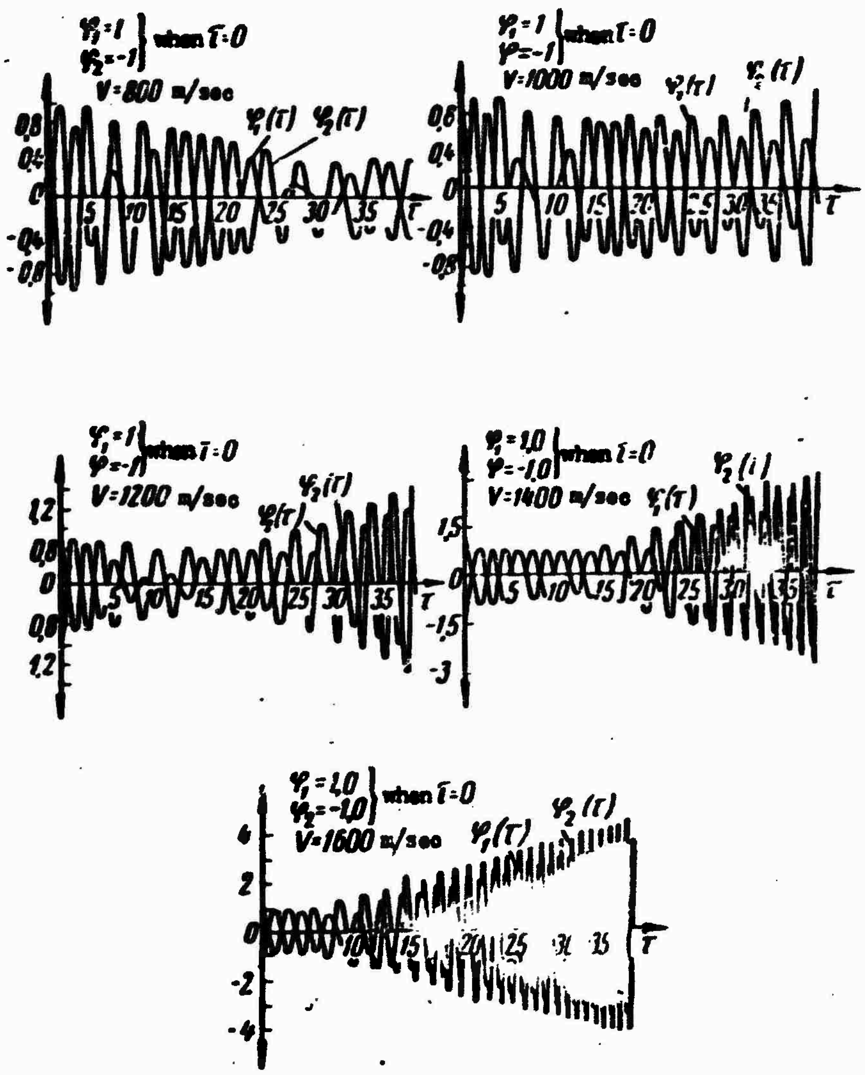



Fig. 18.

to variable $\xi$ according to the law:

$$\tilde{w} = \varphi_1(0)\sin\pi\xi + \varphi_2(0)\sin 2\pi\xi.$$

An analysis of the calculations shows that with a change in the character of amplitudes of the initial deflection of the plate and their increase (9.15a), (9.15b), (9.15c) the frequency of oscillations and the value of the critical speed of the plate increase noticeably. Calculation of nonlinear factors in problems of aerostability of plates for the above-indicated boundary conditions shows that even in the supercritical region no rigid excitation of oscillations is observed; oscillation amplitudes are increasing slowly.

During sufficiently small initial perturbations plate flutter in a nonlinear setting appears at flow rates which differ little from the critical flow rate for the same plate in linear setting. However, with an increase in value and a change in the character of initial deflection of the plate surface, prescribed in the form of the initial sag (9.15), the possibility is revealed of the existence of established motions in flow rates, somewhat exceeding the critical flow rate for a plate in linear setting.

Thus, if the critical flow rate of the plate in linear setting, found by the Bubnov-Galerkin method with a two-term approximation of function of sag with respect to variable $\xi$, constitutes 952 m/sec, then for the same plate in a nonlinear formulation of the problem of aero-elasticity under initial conditions, established by formulas (9.15a), (9.15b), (9.15c), the critical flow rates are equal to 1000, 1050, 1600 m/sec respectively.

The envelope of perturbed solutions of system (9.13) in flow rates, significantly smaller than the critical rate, for all cases of initial conditions (9.15) has the character of a curve, enveloping oscillations rapidly damping in time. With an increase in the flow rate of the envelope, of the curve which outlines the periodic oscillations with a certain increase of oscillation amplitudes at intervals of the first period and only with definite values of flow rates, corresponding in given determination to the critical rates, is the envelope of perturbed solutions $\varphi_1(\tau)$ and $\varphi_2(\tau)$ in the interval of time under consideration and assumes the form of continuously increasing curve for all $\tau > 0$.

The investigation of solutions of a system of differential equations of natural oscillation of plates in nonlinear setting for the class of initial conditions (9.15) and under boundary conditions, which

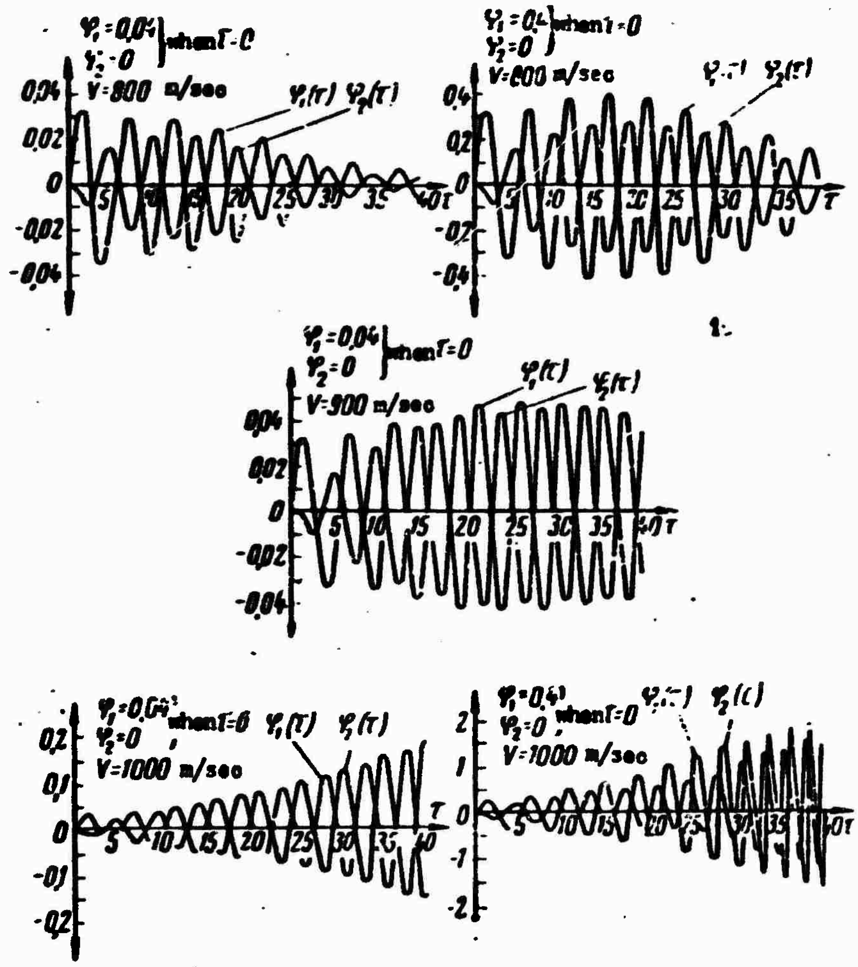



Fig. 19.

present a definite practical interest, show that the sensitivity of a plate to excitation of its flutter sharply decreases with the increase of the initial perturbation, prescribed in the form of the initial sag (9.15).

Of the greatest interest are the results of investigation of perturbed solutions of a system of nonlinear differential equations (9.13), which were conducted for a plate of the same dimensions and with boundary conditions examined earlier, but under initial conditions (9.16). Physically these initial conditions mean that in the moment of time $\tau = 0$ the surface of the plate develops a sag, the rate of variation of which with respect to variable $\xi$ is written in the form

$$\dot{\bar{w}} = \dot{\varphi}_1 \sin \pi\xi.$$

Calculations were performed for values of the maximum initial speed in the center of plate $\varphi_1(0)$, equal to 0.04; 0.4; 0.6.

Results of calculations (Fig. 19) show that a change of maximum initial speed in the center of the plate by one and a half order does not lead to any noticeable change of the value of the critical speed. From the graphs (Fig. 19) it is clear that a change of value of initial perturbation, prescribed in the form of initial speed (9.16a,b), leads only to a change of plate oscillation amplitudes, while the frequency of oscillation of the plate at the initial moment of time remains constant.

## CHAPTER IV

## CERTAIN OTHER DYNAMIC PROBLEMS OF SHELLS

### § 1. Radial Elastic Deformation

If a cylinder of average thickness is under the action of internal pressure p(t), which is uniform along its length, which changes according to the given law, and the pressure front moves along x-axis with a given speed, the problem of exact calculation even of elastic stresses and deformations becomes very complicated with respect to calculations and we cannot find simple formulas, from which we could obtain a clear idea of dynamic effects [41]. In the simplest case of a plane problem the radial and tangential stresses are expressed through $\varepsilon_r = \frac{\partial w}{\partial r}$, $\varepsilon_\theta = \frac{w}{r}$ with the formulas:

$$\sigma_r = 3K\left(\frac{\partial w}{\partial r} + \nu' \frac{w}{r}\right), \quad \sigma_\theta = 3K\left(\frac{w}{r} + \nu' \frac{\partial w}{\partial r}\right), \tag{1.1}$$

where K is the modulus of volume deformation; $\nu' = \frac{\nu}{1 - \nu}$, where $\nu$ is Poisson's ratio. The dynamic equation for radial motion

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 w}{\partial t^2}$$

on the basis of (1.1) is reduced to the form

$$c^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\right) = \frac{\partial^2 w}{\partial t^2}. \tag{1.2}$$

Here $\rho$ is the density of the material, and c is rate of propagation of volume waves

$$c = \sqrt{\frac{3K}{\rho}}. \quad (1.3)$$

General solution of problem with the initial condition

$$t = 0: \quad w = w_0(r), \quad \frac{\partial w}{\partial t} = \dot{w}_0(r)^* \quad (1.3^{I})$$

and boundary conditions

$$\begin{aligned} r = a, \quad & \sigma_r = -p(t); \\ r = b, \quad & \sigma_r = 0 \end{aligned} \quad (1.3^{II})$$

is obtained by the known method by means of substitution

$$w = f(r)e^{i\varkappa ct}.$$

Now equation (1.2) assumes the form

$$f'' + \frac{1}{r}f' + \left(\varkappa^2 - \frac{1}{r^2}\right)f = 0,$$

which is satisfied by function

$$w = [AI_1(\varkappa r) + BN_1(\varkappa r)]e^{i\varkappa ct}, \quad (1.4)$$

where $I_1(\varkappa r)$ and $N_1(\varkappa r)$ are Bessel and Neumann's functions.

Eigenvalues of parameter $\varkappa_n$ (frequencies of free radial oscillations of cylinder $c\varkappa_n$) are found, according to (1.4) and (1.1), from conditions $(1.3^{I,II})$, in which it is assumed that p = 0; the uniform system with respect to A and B results in the frequency equation which has the form [42]

$$D(\varkappa a) = D(\varkappa b), \quad (1.5)$$

where it is designated (when $\nu = 0.25$)

$$D(\varkappa a) = \frac{3\varkappa a N_0(\varkappa a) - 2N_1(\varkappa a)}{3\varkappa a I_0(\varkappa a) - 2I_1(\varkappa a)}. \quad (1.6)$$

---

*From now on the point above the letter designates differentiation with respect to time.

Here $I_0$, $I_1$ are Bessel functions, and $N_0$, $N_1$ are Neumann's functions. The first five roots of equation (1.5) of values of magnitude $(\varkappa b)_n = \left(\frac{\omega a}{\alpha}\right)_n$ for the relation $\alpha = \frac{a}{b} = 0.75$ $\left(\frac{h}{r} \approx 0.3\right)$ are adduced in Table 2.

There are also tables for higher numbers of natural frequencies $(\varkappa b)_n$. Frequency n is equal $c\varkappa_n = (\varkappa b)_n \frac{c}{b}$.

For every root $\varkappa_n$ of the equation (1.5) the relationship of constants A and B becomes fully definite, and formula (1.4) gives an expression for the n-th eigenfunction

$$f_n = -D_n I_1(\varkappa_n r) + N_1(\varkappa_n r). \tag{1.4$^I$}$$

Table 2. Roots of Equation (1.5) for the Ratio $\alpha = a/b = 0.75$

| Number of frequency | $(\varkappa b)_n$ |
|---|---|
| 1 | 1,0865 |
| 2 | 12,6243 |
| 3 | 25,1615 |
| 4 | 37,7183 |
| 5 | 50,2799 |

Now, substituting $\dot{w}_0(r)$, $w_0(r)$ in the form of eigenfunction series from initial conditions $(1.3^I)$ we find values $A_n'$, $A_n''$, i.e., we obtain the solution for the problem on free oscillations for given initial conditions.

For the solution of problem on forced oscillations of a cylinder under the action of pressure p(t) we replace p(t) with the volume radial force q, applied in thin ring $a < 4 \leq r + \delta$, so that boundary conditions become uniform (i.e., $\sigma_r = 0$ when $r = a$ and $r = b$), and equation (1.2) becomes nonhomogenous.

$$c^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2} w\right) + \frac{q}{\rho} = \frac{\partial^2 w}{\partial t^2}. \tag{1.7}$$

Let us select q(r, t) in such a way that when $\delta \to 0$

$$q = p(t)\sum_{n=1}^{\infty} a_n f_n(\varkappa_n r). \tag{1.8}$$

Decomposing q in an eigenfunction series $f_n(\varkappa_n r)$:

$$\int_a^{a+\delta} q\,dr = p(t). \tag{1.8$^I$}$$

the solution of equation (1.7) we present in the form

$$w = \sum_{n=1}^{\infty} \varphi_n(t) f_n(\varkappa_n r). \tag{1.9}$$

where for functions $\varphi_n(t)$ we obtain from (1.7) the system of differential equations:

$$\ddot{\varphi}_n + c^2 \varphi_n \varkappa_n^2 = \frac{a_n}{\rho} p(t),$$

each of which has a particular solution

$$\varphi_n = \frac{a_n}{\rho c \varkappa_n} \int_0^t p(\tilde{\tau}) \sin[c\varkappa_n (t - \tilde{\tau})] d\tilde{\tau}. \tag{1.10}$$

Inasmuch as with $t = 0$ the expression (1.10) gives $\varphi_n = 0$ and $\dot{\varphi}_n = 0$, the (1.9) with values $\varphi_n$ (1.10) formally presents the solution of problem on the action of pressure $p(t)$ on the initially undeformed cylinder.

The series, included in expression ($1.8^{I}$), constitutes an expansion of the discontinuous function $\Delta(r)$:

$$\sum_{n=1}^{\infty} a_n f_n = \Delta(r), \tag{1.11}$$

which may be written in the form

$$\Delta(r) = \begin{cases} \frac{1}{\delta}, & a < r < a + \delta \\ 0, & a + \delta < r. \end{cases} \tag{1.12}$$

Here the function q ($1.8^{I}$) will satisfy condition (1.8), if the final result will have meaning when $\delta \to 0$.

Considering the orthogonality of functions $f_n$,

$$\int_a^b r f_n f_m dr = \begin{cases} 0, & m \neq n, \\ c_n^2, & m = n, \end{cases} \tag{1.13}$$

multiplying by $rf_m dr$ both parts of (1.11) and integrating from a to b, we will obtain:

$$a_n = \frac{a}{c_n^2} f_n(\varkappa_n a), \quad c_n^2 = \int_a^b f_n^2 r dr. \tag{1.14}$$

This means, that the general solution is fully determined by formula (1.9), which will be written in the form

$$w = \frac{a}{\rho c} \sum_{n=1}^{\infty} \frac{f_n(\varkappa_n a)}{\varkappa_n c_n^2} f_n(\varkappa_n r) \int_0^t p(\tilde{\tau}) \sin[c\varkappa_n (t - \tilde{\tau})] d\tilde{\tau}. \tag{1.15}$$

This expression is too complicated for analysis and requires bulky numerical calculations. In the simplest case, when in the moment $t = 0$ constant pressure $p = \text{const}$ is applied, formula (1.15) assumes this form (taking into consideration that $\rho c^2 = 3k$)

$$w = \frac{pa}{3k} \sum_{n=1}^{\infty} \frac{1}{c_n^2 \varkappa_n^2} f_n(\varkappa_n a) f_n(\varkappa_n r) [1 - \cos(c\varkappa_n t)]. \tag{1.16}$$

The coefficient of dynamics factor, showing the ratio of displacements (and stresses) in dynamic calculation of cylinder (taking into account forces of inertia of substance) to their values in static calculation, depends strongly on the law of application of pressure $p(t)$ and can not only fail to take value 2, but also to be essentially less than unity in the case of brief actions. Cylinders can sustain, while remaining elastic, pressures exceeding many times the maximum permissible static pressures, if the time of action of the pressure is less than the time of double passage of sonic wave $2\frac{b - a}{c}$ through the wall thickness, which fact is essential and should be taken into consideration for very thick-walled cylinders. This effect, consequently, first of all, pertains to large elastic masses with cylindrical cavities [41] and therefore is not examined here.

## § 2. Plane Elastoplastic Deformation

The dynamic problem for the cylinder in the case of plane elastoplastic deformation is somewhat simplified, inasmuch as the cylinder may be considered to be a mechanical system with one degree of freedom [41].

Let us assume that when $t = 0$ the cylinder with radii a, b is at

rest, but when $t > 0$ the internal pressure $p_a(t)$ and external $p_b(t)$ pressure, act so that the initial coordinate of any particle $r_0$ changes to value $r(r_0, t)$, and the internal radius a becomes equal to $R(t)$. Let us assume that $v = \frac{dR}{dt}$ is the rate of expansion of the cavity. The condition of incompressibility of material:

$$r^2 - R^2 = r_0^2 - a^2, \quad R_b^2 - R^2 = b^2 - a^2 \tag{2.1}$$

enables us for small and ultimate deformations to write expressions of shear $\gamma$, rate $v_r = \frac{dr}{dt}$, and acceleration $\frac{dv_r}{dt}$:

$$\gamma = \frac{r}{r_0} - \frac{\partial r}{\partial r_0} = \frac{R^2 - a^2}{r\sqrt{r^2 - R^2 + a^2}},$$

$$v_r = \frac{R}{r} v, \quad \frac{dv_r}{dt} = \frac{1}{r} \frac{d}{dt}(Rv) - \frac{R^2 v^2}{r^3}. \tag{2.2}$$

The dynamic equation in the case of ultimate strains is written in the form

$$\frac{\partial \sigma_r}{\partial r} = \frac{\rho}{r}\left[\frac{d}{dt}(Rv) - \frac{R^2 v^2}{r^2}\right] + \frac{2\tau(\gamma)}{r},$$

where $\tau = F(\gamma)$ is the material strengthening function, which on the basis of (2.2) is expressed through R and r.

Integrating this equation with respect to r from the internal surface $(r = R)$ to external $R_b = r$ surface and taking into consideration boundary conditions, we obtain:

$$\frac{dv}{dt} R \ln \frac{R_b}{R} + \frac{v^2}{2}\left(\ln \frac{R_b}{R^2} + \frac{R^2}{R_b^2} - 1\right) + \frac{2}{\rho}\int_R^{R_b} \frac{\tau}{r} dr = \frac{p}{\rho}, \tag{2.3}$$

where $p = p_a - p_b$ is the difference of pressures, which determines the motion.

In the beginning let us examine small elastic deformations. In this case, designating $w(t) = R(t)$ a and discard in (2.2) small values of the order of $\frac{w}{a}$ with respect to 1, we obtain ($R_b = b$, $R_a = a$):

$$\gamma = \frac{2aw}{r^2}, \quad \tau = G\gamma.$$

In equation (2.3) in addition to this simplification it is also necessary to discard small values of the order of $v^2$ with respect to $\frac{p}{\rho}$ and $\frac{\tau}{\rho}$, after which, designating the wave velocity of the shear $c_1$ and parameter $\varkappa$ according to formulas:

$$c_1=\sqrt{\frac{G}{\rho}},\quad \varkappa=\frac{1}{b}\sqrt{\frac{2(b^2-a^2)}{a\ln\frac{b}{a}}}, \tag{2.4}$$

we obtain the equation in the form

$$\frac{d^2w}{dt^2}+c_1^2\varkappa^2w=\frac{p(t)}{\rho a\ln\frac{b}{a}}, \tag{2.5}$$

the solution of which, analogously to (1.10), will be

$$w=\frac{1}{c_1\varkappa\rho a\ln\frac{b}{a}}\int_0^t p(\tilde{\tau})\sin[c_1\varkappa(t-\tilde{\tau})]\,d\tilde{\tau}. \tag{2.6}$$

The dynamics factor $k_0$ can be determir ³ with the definite degree of authenticity on the basis of the solution (2.6).

Let us assume that p(t) has with a certain $t = t_m$ a maximum $p_m$; the static estimate for the action of pressure $p_m$ yeilds the value of maximum tangetial stress

$$\tau_m=\frac{1}{1-a^2}p_m.$$

The dynamic estimate gives for r = a

$$\tau=2G\frac{w}{a},$$

while w is determined according to (2.6). This means, that the dynamics factor $k_\partial$ is determined as the value biggest in time with respect to the modulus of ratio $\frac{\tau}{\tau_m}$, i.e.,

[$\partial$ = d = dynamic]

$$k_\partial=\max\left|\frac{c_1\varkappa}{p_m}\int_0^t p(\tilde{\tau})\sin[c_1\varkappa(t-\tilde{\tau})]\,d\tilde{\tau}\right|. \tag{2.7}$$

For the constant pressure appearing instantly

$$p(t)=\begin{vmatrix}0, & t<0\\ p=\text{const}, & t>0\end{vmatrix}$$

from (2.7) we obtain:

$$k_0 = \max \left| \int_0^t \sin(t-\tilde{\tau})\, d\tilde{\tau} \right| = 2.$$

In the case of constant pressure effective in a definite interval of time $t_1$:

$$p(t) = \begin{cases} 0, & t<0,\ t>t_1 \\ p, & 0<t<t_1, \end{cases}$$

the dynamics factor will be equal to the biggest of the expressions:

$$k_0' = \max \left| \int_0^t \sin(t-\tilde{\tau})\, d\tilde{\tau} \right| = \max(1-\cos t),\ t<t_1,$$

$$k_0'' = \max \left| \int_0^{t_1} \sin(t-\tilde{\tau})\, d\tilde{\tau} \right| = \max\left|\cos(t-t_1) - \cos t\right|,\ t>t_1,$$

where $t_1 = c_1 \varkappa \tilde{\tau}$.

Now we will consider elastoplastic small deformations. In one-sided dynamic process (expansion or compression) the dependency $\tau = F(\gamma)$ allows us to express the integral included in (2.3), through w. Presenting $F(\gamma)$ in the form $\tau = G\gamma[1 - \omega(\gamma)]$, we obtain:

$$2\int_a^b \frac{\tau\, dr}{r} = \int_{\gamma_b}^{\gamma_a} \frac{\tau}{\gamma}\, d\gamma = 2G(1-a^2)\frac{w}{a} - 2G\int_{\gamma_b}^{\gamma_a} \omega(\gamma)\, d\gamma.$$

Designating,

$$\gamma_a = \frac{2w}{a}, \quad \gamma_b = \frac{2w}{a}a^2, \quad S = \frac{2c_1^2}{\rho a \ln\frac{1}{a}}, \tag{2.8}$$

we convert the dynamic equation (2.3) to form

$$\frac{d^2 w}{dt^2} + (c_1\varkappa)^2 w - s\int_{\gamma_b}^{\gamma_a} \omega(\gamma)\, d\gamma = \frac{p(t)}{\rho a \ln\frac{1}{a}}. \tag{2.9}$$

This quasi-linear differential equation has a small parameter, inasmuch as function $\omega < 1$. Therefore, the solution can be found by the method of small parameter, for which it is possible to take s. Here we examine the monotonous solution w, either increasing or diminishing in time. Therefore, following the method of elastic solutions, for the first approximation we must take the solution of

elastic problem (2.6). The second approximation is also obtained by the formula (2.6), if in it p(t) is replaced by

$$p(t) + 2c_1^2 \int_{\gamma_{b_1}}^{\gamma_{a_1}} \omega(\gamma)\, d\gamma.$$

where

$$\gamma_{b_1} = \gamma_{a_1}\alpha^2$$

and

$$\gamma_{a_1} = \frac{2}{c_1 \varkappa \rho a^2 \ln \frac{1}{\alpha}} \int_0^t p(\tilde{\tau}) \sin [(c_1 \varkappa (t - \tilde{\tau})]\, d\tilde{\tau}. \tag{2.10}$$

Thus, in the second approximation we obtain:

$$\omega = \frac{1}{c_1 \varkappa \rho a \ln \frac{1}{\alpha}} \int_0^t \left[ p(\tilde{\tau}) + 2c_1 \int_{\gamma_{b_1}(\tilde{\tau})}^{\gamma_{a_1}(\tilde{\tau})} \omega(\gamma)\, d\gamma \right] \sin [c_1 \varkappa (t - \tilde{\tau})]\, d\tilde{\tau}. \tag{2.11}$$

Let us examine large plastic deformations. If pressure p(t) = = $p_a - p_b$ depends only on the volume of the cylinder's cavity, i.e., p = p(R), then equation (2.3) has the energy integral, and is linear with respect to $\frac{v^2}{2}$:

$$R \ln \frac{R_b}{R} \frac{d}{dR}\left(\frac{v^2}{2}\right) + \left(\ln \frac{R_b^2}{R^2} + \frac{R^2}{R_b^2} - 1\right)\frac{v^2}{2} = $$
$$= \frac{p(R)}{\rho} - \frac{2}{\rho}\int_R^{R_b} \frac{\tau dr}{r}. \tag{2.12}$$

Disregarding elastic deformations and material strengthening, i.e., assume that $\tau = \frac{\sigma_s}{\sqrt{3}} = \tau_s$, we write the integral of equation (2.12) from the condition of energy conservation. The internal pressure $p_s$, necessary for surmounting plastic resistance of material, in this case is equal to

$$p_s = \frac{\sigma_s}{\sqrt{3}} \ln \frac{R_b^2}{R^2}. \tag{2.13}$$

Thus, part of the effective pressure p(R), which will increase the kinetic energy of the cylinder, will be equal to p(R) - $p_s$(R). The

corresponding work will be

$$A = 2\pi \int_a^R \left[ p(R) - \frac{\sigma_s}{\sqrt{3}} \ln \frac{R_b^2}{R^2} \right] R\,dR. \tag{2.14}$$

The kinetic energy of a cylinder on the basis of (2.2) will be written thus:

$$T = \pi\rho \int_R^{R_b} v_r^2 r\,dr = \pi\rho R^2 v^2 \ln \frac{R_b}{R}. \tag{2.15}$$

From the law of conservation of energy we have:

$$T = T_0 + A, \tag{2.16}$$

where $T_0$ is the initial kinetic energy of the cylinder

$$T_0 = \pi\rho a^2 v_0^2 \ln \frac{b}{a}. \tag{2.17}$$

The rate of expansion of the cylinder's internal surface from (2.16) is found in the form of function of radius R:

$$v = \frac{1}{R} \sqrt{\frac{T_0 + A}{\pi\rho \ln \frac{R_b}{R}}}. \tag{2.18}$$

If $T_0 = 0 (v_0 = 0)$ and pressure p decreases with the expansion of the cavity, so that beginning with certain R the expression in brackets under integral (2.14) becomes negative, the rate v will have the maximum $v_m$. Designating with $R_m (R_{mb})$ the radius, with which the maximum rate is attained, we obtain the relationship between $v_m$ and $R_m$ from equation (2.12), in which we must assume that

$$\frac{d}{dR}\left(\frac{v^2}{2}\right) = 0. \tag{2.19}$$

Radius $R_m$ is found from equation

$$\frac{d}{dR}\left\{ \frac{1}{\ln \frac{R_b}{R}} \int_a^R \left[ p(R) - \frac{\sigma_s}{\sqrt{3}} \ln \frac{R_b^2}{R^2} \right] R\,dR \right\} = 0, \qquad (R = R_m). \tag{2.20}$$

Let us consider the particular case of compression of the cylinder's cavity at the expense of initial kinetic energy $T_0$. Assuming that

in (2.14) p = 0, we find:

$$A = \frac{\pi \sigma_s}{\sqrt{3}}\left[R^2 \ln \frac{R_b^2}{R^2} - a^2 \ln \frac{b^2}{a^2} + (b^2 - a^2) \ln \frac{R_b^2}{b^2}\right]. \tag{2.21}$$

The movement according to (2.16) will cease with the R, determined from condition $T_0 + A = 0$. We find the least kinetic energy $T_{cr}$, when the cavity will be closed. Passing to limit $R \to 0$, we obtain:

[kp = cr = critical]

$$T_{\kappa p} = \frac{\pi \sigma_s}{\sqrt{3}}\left[a^2 \ln \frac{b^2}{a^2} + (b^2 - a^2) \ln \frac{b^2}{a^2}\right]. \tag{2.22}$$

and the corresponding initial rate will be found from equation

$$\frac{\rho \dot{R}^2_{\kappa p}}{2} = \frac{T_{\kappa p}}{\pi a^2 \ln \frac{b^2}{a^2}}. \tag{2.23}$$

When $T_0 > T_{cr}$ the cavity will be slammed closed with the speed, which with $R \to 0$ tends toward infinity:

$$\frac{\rho \dot{R}^2}{2} \sim \frac{T_0 - T_{\kappa p}}{\pi R^2 \ln\left(\frac{b^2 - a^2}{R^2}\right)}. \tag{2.24}$$

## § 3. Action of a Moving Load on a Cylinder

It is possible to give an estimate of the dynamics factor with a mobile load, on the basis of the theory of oscillations of a cylindrical shell [41].

We give the equation of radial oscillations of the cylindrical shell:

$$\rho h \frac{\partial^2 w}{\partial t^2} + D \frac{\partial^4 w}{\partial x^4} + \left(\frac{2\nu D}{R^2} - \frac{P}{2\pi R}\right) \frac{\partial^2 w}{\partial x^2} + \frac{12(1-\nu^2) D}{h^2 R^2} w = p - \frac{\nu P}{2\pi R^2}. \tag{3.1}$$

Here: h is thickness of wall, R is radius, $D = \dfrac{Eh^3}{12(1 - \nu^2)}$ is cylindrical rigidity, P is constant axial stretching force, p is internal pressure.

Let us examine forced oscillations of the cylinder under the action of constant pressure p and annular pressure Q = qc, moving with constant speed v along the cylinder to the right, where we shall assume that pressure on the right of Q is equal to zero (Fig. 20).

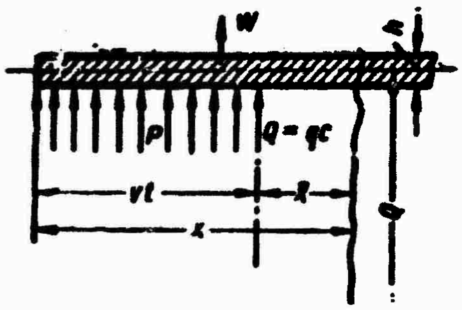


Fig. 20.

The stationary solution, constant in axes, moving together with the load is of interest. Let us assume that the origin of coordinates is at the point of application of annular pressure Q. We examine, consequently, the solution of equation (3.1), depending on the difference

$$\xi = x - vt, \tag{3.2}$$

then

$$\frac{\partial}{\partial x} = \frac{d}{d\xi}; \quad \frac{\partial}{\partial t} = -v\frac{d}{d\xi}.$$

and therefore, (3.1) assumes the form

$$Dw^{IV} + \left(\rho h v^2 + \frac{2\nu D}{R^2} - \frac{P}{2\pi R}\right)w^{II} + \frac{12(1-\nu^2)D}{h^2R^2}w = p - \frac{\nu P}{2\pi R^2}, \tag{3.3}$$

where a stroke signifies the derivative with respect to $\xi$. From the comparison of the first and last component of the right side (3.3) it is clear that the characteristic size of the region of variation of deformations will be of the order of

$$l = \frac{\sqrt{Rh}}{\sqrt[4]{3(1-\nu^2)}}. \tag{3.4}$$

and therefore, in the order of values, the equation (3.3) has the form

$$D\frac{w}{l^4} + \left(\rho h v^2 + \frac{2\nu D}{R^2} - \frac{P}{2\pi R}\right)\frac{w}{l^2} + D\frac{w}{l^4} \approx p.$$

The ratio of the second component to the first and third is determined by values

$$\frac{\rho h v^2 \frac{w}{l^2}}{D\frac{w}{l^4}}; \quad \frac{\frac{2\nu D}{R^2}\frac{w}{l^2}}{D\frac{w}{l^4}}; \quad \frac{\frac{P}{2\pi R}\frac{w}{l^2}}{D\frac{w}{l^4}}.$$

With very large rates v the first ratio may be large, and therefore, it cannot be disregarded. The second ratio equals

$$2\nu \frac{l^2}{R^2} = \frac{2\nu}{\sqrt{3(1-\nu^2)}}\frac{h}{R},$$

i.e., the corresponding term in (3.3) can be disregarded with an error not exceeding $\frac{h}{R}$ as compared with unity. The third ratio is equal to

[cp = av = average] $$\frac{Pl^2}{2\pi RD} = \frac{P}{2\pi Rh}\frac{l^2h12}{Eh^3}(1-\nu^2) = 4\sqrt{3(1-\nu^2)}\frac{R}{h}\frac{(\sigma_3)_{cp}}{E},$$

where $(\sigma_3)_{av} = \frac{P}{2\pi Rh}$ is the average stretching stress from axial force P; the ratio shown is a minute value, and the corresponding component in parentheses (3.3) can be disregarded.

Thus, the dynamic equation (3.3) has the following approximate form:

$$l^4\frac{d^4w}{d\xi^4} + \rho\frac{hv^2l^2}{D}\frac{d^2w}{d\xi^2} + 4w = \frac{l^4}{D}\left(p - \frac{\nu P}{2\pi R^2}\right). \qquad (3.5)$$

We introduce new designations, simplifying the formulation of the problem, namely, the dimensionless coordinate:

$$\zeta = \frac{\xi}{l} = \xi\frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{Rh}}, \qquad (3.6)$$

the static sag of shell according to the zero-moment theory

[ct = st = static] $$w_{ст} = \frac{1}{4}\frac{l^4}{D}\left(p - \frac{\nu P}{2\pi R^2}\right) \qquad (3.7)$$

and dynamics parameter:

$$\chi = \frac{\rho h v^2 l^2}{4D} = \sqrt{3(1-\nu^2)}\frac{R}{h}\left(\frac{v}{c}\right)^2, \qquad (3.8)$$

$$c = \sqrt{\frac{E}{\rho}},$$

where c is the transonic speed in material. From (3.5) we obtain:

$$\frac{d^4w}{d\zeta^4} + 4\chi\frac{d^2w}{d\zeta^2} + 4w = 4w_{ст}. \qquad (3.9)$$

Since $w_{st}$ is a function, which has a break in one point x = 0, then, dividing the region into intervals x < 0 and x > 0, we will obtain for

$x < 0$ $w = w_{st}$. The general solution in the form $e^{k\zeta}$ leads to the characteristic equation

$$k^4 + 4\chi k^2 + 4 = 0,$$

having roots:

$$\begin{gathered} k_{1,2} = s_1 \pm is, \\ k_{3,4} = -s_1 \pm is, \\ s_1 = \sqrt{1-\chi}, \quad s = \sqrt{1+\chi}. \end{gathered} \tag{3.10}$$

The sag w for the left-hand part of cylinder $x < 0$ has the form

[л = l = left] $$w = w_л + e^{s_1\zeta}(A_1 \cos s\zeta + B_1 \sin s\zeta), \tag{3.11}$$

and for the right-hand part ($x > 0$)

[п = r = right] $$w = w_п + e^{-s_1\zeta}(A_2 \cos s\zeta + B_2 \sin s\zeta), \tag{3.12}$$

where A, B are arbitrary constants and

$$w_л - w_п = \frac{pR^2}{Eh} = \tilde{p}. \tag{3.13}$$

Conditions of conjugation of solutions in cross-section $x = 0$ require a continuity of sag, and angle of inclination of generator, a bending moment and intersecting force $N = \frac{dM}{dx}$, which in cross section $x = 0$ should have a break by a value Q. Designating

$$\tilde{q} = \frac{Qr^3}{D} = 4\sqrt[4]{3(1-\nu^2)}\,\frac{qcR}{Eh}\sqrt{\frac{R}{h}} \tag{3.14}$$

and satisfying conditions of conjugation, we find constants in (3.11) and (3.12):

$$\begin{gathered} A_1 = -\frac{1}{2}\tilde{p} + \frac{1}{8s_1}\tilde{q}, \quad B_1 = -\frac{1}{2}\tilde{p}\frac{\chi}{\sqrt{1-\chi^2}} - \frac{1}{8s}\tilde{q}, \\ A_2 = \frac{1}{2}\tilde{p} + \frac{1}{8s_1}\tilde{q}, \quad B_2 = -\frac{1}{2}\tilde{p}\frac{\chi}{1-\chi^2} + \frac{1}{8s}\tilde{q}. \end{gathered} \tag{3.15}$$

From (3.11), (3.12) and (3.13) it is clear that $\chi = 1$ determines the critical speed of motion of the load, with which a strong influence of

the dynamic load is possible. When $\chi = 1$ we obtain from (3.8):

$$v_{\text{кр}} = \frac{c}{\sqrt{3(1-\nu^2)}}\sqrt{\frac{h}{R}} = 4\cdot 10^{3}\sqrt{\frac{h}{R}}\ \frac{\text{м}}{\text{сек}} \tag{3.16}$$

(the number is given for $E = 2.1\cdot 10^6$ kg/cm$^2$, $\nu = \frac{1}{3}$, $\rho g = 7.8$).

When $\chi < 1$ the sag at the expense of the load dynamics will be larger than with its static application ($\chi = 0$), and therefore, we can determine the dynamics factor as the ratio of maximum sag $w_{max}$ when $\chi > 0$ to the maximum sag when $\chi = 0$:

$$k_{\partial} = \frac{(w_{max})_{\chi>0}}{(w_{max})_{\chi=0}}, \tag{3.17}$$

where $w_{max}$ is determined for the left-hand part of the cylinder, i.e., by the formula (3.11). Point $\zeta_{max} < 0$, in which sag is the biggest, is determined from the condition $\frac{dw}{d\zeta} = 0$ which gives

$$\operatorname{tg}(s\zeta_{max}) = \frac{\sqrt{1+\chi}}{\sqrt{1-\chi} - \frac{\tilde{q}}{2\tilde{p}}}. \tag{3.18}$$

Let us examine the first example, when annular pressure Q spreads at the rate v, so that $\tilde{p} = 0$, $\tilde{q} \neq 0$. The maximum dynamic sag will be

$$w_{max} = A_1 = \frac{\tilde{q}}{8\sqrt{1-\chi}},$$

and therefore, the dynamics factor is equal to $k_d = \frac{1}{\sqrt{1-\chi}}$.

We consider the second example, when the axial force and annular load are absent ($P = Q = 0$, $\tilde{q} = 0$) and only the internal pressure is active. The biggest sag is obtained in point $\zeta_{max} < 0$, for which

$$\operatorname{tg}(s\zeta_{max}) = \sqrt{\frac{1+\chi}{1-\chi}},$$

so that, if we designate

$$\frac{\pi}{2} < \Delta = \operatorname{arctg}\left(-\sqrt{\frac{1+\chi}{1-\chi}}\right) < \frac{3\pi}{4} = \Delta_0,$$

then for the maximum sag we obtain the expression

$$w_{max} = \frac{\tilde{p}}{2\sqrt{\xi}}\left[2\sqrt{\xi} + e^{-\Delta} \times \right.$$
$$\left. \times\left(\sqrt{1-\chi} + \frac{\chi}{1-\chi}\sqrt{1+\chi}\right)\right]. \qquad (3.19)$$

and therefore, the dynamics factor

$$k_d = \frac{2\sqrt{\xi} + e^{-\Delta}\left(\sqrt{1-\chi} + \frac{\chi}{1-\chi}\sqrt{1+\chi}\right)}{2\sqrt{\xi} + e^{-\Delta_0}}. \qquad (3.20)$$

The dynamics factor of the annular pressure turns out to be significantly larger than the dynamics factor of internal pressure. For instance, when $\chi = 0.45$ we have $K_{d\tilde{q}} = 1.35$, $K_{d\tilde{p}} = 1.035$; for shells with the relationship $\frac{h}{R} \approx 0.2$ this corresponds to the load speed $v \approx 1200$ m/sec.

## § 4. On the Propagation of Elastic Waves in a Shell

Of interest is the research on propagation of perturbations in thin-walled structures in connection with the problem of their durability and rigidity. Let us consider propagation of elastic waves in shells.

In the beginning let us write the system of differential equations [43] for symmetric oscillations of the circular cylindrical shell with the thickness of 2h and radius of the middle surface $r_0$, which will then be used in the study of propagation of elastic waves in a shell.

Proceeding from general equations of the theory of elasticity without any hypotheses about the character of deformation, on the basis of N. A.Kil'chevskiy's [44] algorithm, by excluding from matrix operators all displacement functions, with the exception of one or several, I. T. Selezov constructed generalized differential oscillation

equations* [45]:

$$
\begin{gathered}
\left\{[\xi a_0+\xi^3 a_1]-\xi^3 a_2\frac{\partial^2}{\partial x^{*2}}+[\xi+\xi^3 a_3]\frac{\partial^2}{\partial t^{*2}}+\xi^3 a_4\frac{\partial^4}{\partial x^{*4}}-\right.\\
\left.-\xi^3 a_5\frac{\partial^4}{\partial t^{*2}\partial x^{*2}}+\xi^3 a_6\frac{\partial^4}{\partial t^{*4}}\right\}w_0^*+\left\{[\xi a_7+\xi^3 a_8]\frac{\partial}{\partial x^*}+\right.\\
\left.+\xi^3 a_9\frac{\partial^3}{\partial x^{*3}}+\xi^3 a_{10}\frac{\partial^3}{\partial t^{*2}\partial x^*}\right\}u_0^*=\left\{[2+\xi^3 d_1]-\xi^3 d_2\frac{\partial^2}{\partial x^{*2}}+\right.\\
\left.+\xi^3 d_3\frac{\partial^2}{\partial t^{*2}}\right\}\frac{q_1^*-q_2^*}{2}+\left\{[\xi+\xi^3 d_4]\frac{\partial}{\partial x^*}+\xi^3 d_5\frac{\partial^3}{\partial x^{*3}}+\right.\\
\left.+\xi^3 d_6\frac{\partial^3}{\partial t^{*2}\partial x^*}\right\}\frac{p_1^*+p_2^*}{2}+\left\{[\xi d_7+\xi^3 d_8]-\xi^3 d_9\frac{\partial^2}{\partial x^{*2}}+\right.\\
\left.+\xi^3 d_{10}\frac{\partial^2}{\partial t^{*2}}\right\}\frac{q_1^*+q_2^*}{2};
\end{gathered}
\tag{4.1}
$$

$$
\begin{gathered}
\left\{[-\xi b_0+\xi^3 b_1]\frac{\partial^2}{\partial x^{*2}}+[\xi+\xi^3 b_2]\frac{\partial^2}{\partial t^{*2}}+\xi^3 b_3\frac{\partial^4}{\partial x^{*4}}-\right.\\
\left.-\xi^3 b_4\frac{\partial^4}{\partial t^{*2}\partial x^{*2}}+\xi^3 b_5\frac{\partial^4}{\partial t^{*4}}\right\}u_0^*+\\
+\left\{[-\xi b_6-\xi^3 b_7]\frac{\partial}{\partial x^*}-\xi^3 b_8\frac{\partial^3}{\partial x^{*3}}-\xi^3 b_9\frac{\partial^3}{\partial t^{*2}\partial x^*}\right\}w_0^*=\\
=\left\{[2+\xi^3 l_1]-\xi^3 l_2\frac{\partial^2}{\partial x^{*2}}+\xi^3 l_3\frac{\partial^2}{\partial t^{*2}}\right\}\frac{p_1^*-p_2^*}{2}+\\
+\left\{[\xi l_4+\xi^3 l_5]\frac{\partial}{\partial x^*}-\xi^3 l_6\frac{\partial^3}{\partial x^{*3}}+\xi^3 l_7\frac{\partial^3}{\partial t^{*2}\partial x^*}\right\}\frac{q_1^*+q_2^*}{2}+\\
+\left\{[\xi+\xi^3 l_8]-\xi^3 l_9\frac{\partial^2}{\partial x^{*2}}+\xi^3 l_{10}\frac{\partial^2}{\partial t^{*2}}\right\}\frac{p_1^*+p_2^*}{2}.
\end{gathered}
\tag{4.2}
$$

where $x^*$ is the variable, counted off along the shell axis $w_0^*$ and $u_0^*$ are the radial and axial displacement, $q_{1,2}^*$ is the radial load on internal and external surfaces of the shell, $p_{1,2}^*$ is the axial load on the same surfaces, and $a_j$ and $b_j$ are constants, depending on Poisson's number $\frac{1}{\nu}$. Here the following dimensionless parameters are taken:

$$
w_0^*=\frac{w_0}{r_0},\quad u_0^*=\frac{u_0}{r_0},\quad x^*=\frac{x}{r_0},\quad t^*=\frac{c_s}{r_0}t,
$$
$$
q^*=\frac{q}{\mu c^2},\quad p^*=\frac{p}{\mu c_s^2},\quad \xi=\frac{2h}{r_0},\quad c_s^2=\frac{\mu}{\rho}.
$$

---

*With the accuracy up to terms of the order of $\xi^3$.

Here, as the author [45] thinks, the developed method leads to preservation in differential equations of all terms up to a definite order of smallness and presence of all possible partial derivatives, and therefore, he is of the opinion that the limits of applicability of equations are determined by the order of remaining terms.*

We investigate the infinite cylindrical shell, to which at a distance $x^{*} = 0$ a concentrated axisymmetric momentum $Q\sigma(t^{*})$ is applied, where $\sigma(t^{*})$ is Keaviside's function. We will solve the problem, excluding points of application of concentrated momentum. Then we arrive at differential equations of the form:

$$\begin{aligned}
&\Big\{[(-b_0a_0+a_7b_0)+\xi^2(-a_1b_0+b_1a_0+a_6b_6+a_7b_7)]\frac{\partial^8}{\partial x^{*8}}+\\
&+[a_0+\xi^2(a_1+b_5a_0)]\frac{\partial^2}{\partial t^{*2}}+\xi^2[a_2b_0+b_5a_0+a_7b_8+a_6b_6]\frac{\partial^6}{\partial x^{*6}}+\\
&+[-b_0+\xi^2(-b_0a_2-a_2-b_6a_0+b_1+a_7b_9+a_{10}b_0)]\frac{\partial^4}{\partial t^{*2}\partial x^{*2}}+\\
&+[1+\xi^2(a_3+b_5a_0+b_2)]\frac{\partial^4}{\partial t^{*4}}-\xi^2b_0a_4\frac{\partial^6}{\partial x^{*6}}+\\
&+\xi^2[b_0a_3+a_4+b_3]\frac{\partial^6}{\partial t^{*2}\partial x^{*4}}-\xi^2[b_0a_4+a_5+b_4]\frac{\partial^6}{\partial t^{*4}\partial x^{*2}}+\\
&+\xi^2[a_6+b_5]\frac{\partial^6}{\partial t^{*6}}\Big\}w_0^{*}=0; \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
&\Big\{[-b_6+\xi^2b_1]\frac{\partial^2}{\partial x^{*2}}+[1+\xi^2b_2]\frac{\partial^2}{\partial t^{*2}}+\xi^2b_3\frac{\partial^4}{\partial x^{*4}}-\\
&-\xi^2b_4\frac{\partial^4}{\partial t^{*2}\partial x^{*2}}+\xi^2b_5\frac{\partial^4}{\partial t^{*4}}\Big\}u_0^{*}+\\
&+\Big\{[-b_6-\xi^2b_7]\frac{\partial}{\partial x^{*}}-\xi^2b_8\frac{\partial^3}{\partial x^{*3}}-\xi^2b_9\frac{\partial^3}{\partial t^{*2}\partial x^{*}}\Big\}w_0^{*}=0. \qquad (4.4)
\end{aligned}$$

These equations are obtained from (4.1) and (4.2), and terms of the order higher than $\xi^3$ are rejected.

We assume in points of application of concentrated momentum condition of conjugation are fulfilled, which ensue from general conditions

*The method needs a strict mathematical foundation.

of continuity of displacements and deformations:

$$
\begin{gathered}
\left.\frac{\partial \dot{w}_0}{\partial x^*}\right|_{+0} = 0, \\
\left.\frac{\partial^2 \dot{w}_0}{\partial x^{*2}}\right|_{+0} - \left.\frac{\partial^2 \dot{w}_0}{\partial x^{*2}}\right|_{-0} = -\frac{1}{\xi^2}\chi\frac{\dot{q}_1 - \dot{q}_2}{2}, \\
\left.\dot{u}_0\right|_{+0} - \left.\dot{u}_0\right|_{-0} = 0.
\end{gathered}
\tag{4.5}
$$

Three conditions are satisfied at infinity. Here $\chi = \frac{24(2\lambda + 3\mu)}{\lambda + 2\mu}$, where $\lambda$ and $\mu$ are Lamé constants. In moment $t^* = 0$, if we accept the zero initial conditions, such initial conditions are fulfilled, which also ensue from the exact formulation of the problem:

$$
\begin{gathered}
\dot{w}_0 = \frac{\partial \dot{w}_0}{\partial t^*} = \frac{\partial^2 \dot{w}_0}{\partial t^{*2}} = \frac{\partial^3 \dot{w}_0}{\partial t^{*3}} = 0, \\
\dot{u}_0 = \frac{\partial \dot{u}_0}{\partial t^*} = \frac{\partial^2 \dot{u}_0}{\partial t^{*2}} = \frac{\partial^3 \dot{u}_0}{\partial t^{*3}} = 0.
\end{gathered}
\tag{4.6}
$$

The solution of equations (4.3) and (4.4) in the Laplace image space, satisfying conjugation conditions (4.5), conditions at infinity and initial conditions (4.6), has the form:

$$
\begin{aligned}
\frac{2\xi^3}{\chi Q} W_0(x^*, p) = \sum_{k=1,2} \frac{A_k \exp(-n_k |x^*|)}{P_k^* [(n_1^2 - n_3^2) A_1 + (n_3^2 - n_2^2) A_2]} - \\
- \frac{(A_1 + A_2)\exp(-n_3 |x^*|)}{pn_3 [(n_1^2 - n_3^2) A_1 + (n_3^2 - n_2^2) A_2]}.
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\frac{2\xi^3}{\chi Q} U(x^*, p) = -\sum_{k=1,2} \frac{l_k A_k \exp(-n_k |x^*|)}{pg_k [(n_1^2 - n_3^2) A_1 + (n_3^2 - n_2^2) A_2]} + \\
+ \frac{l_3 (A_1 + A_2)\exp(-n_3 |x^*|)}{pg_3 [(n_1^2 - n_3^2) A_1 + (n_3^2 - n_2^2) A_2]}.
\end{aligned}
\tag{4.8}
$$

where $n_k (k = 1, 2, 3)$ are roots of characteristic equation Re $n_k > 0$,

$$
W_0(x^*, p) = \int_0^\infty e^{-pt^*} \dot{w}_0(x^*, t^*) dt^*,
$$

$$
U_0(x^*, p) = \int_0^\infty e^{-pt^*} \dot{u}_0(x^*, t^*) dt^* -
$$

of Laplace transformation,

$A_1 = g_1(l_2 g_3 - l_3 g_2)$, $A_2$ is obtained from $A_1$ by cyclic permutation of indices:

$$l_i = \xi^2 b_9 n_i^2 + [(b_6 + \xi^2 b_7) + \xi^2 b_8 p^2],$$

$$g_i = \xi^2 b_5 n_i^4 - [(b_6 - \xi^2 b_1) + \xi^2 b_4 p^2] n_i^2 + [(1 + \xi^2 b_3) p^2 + \xi^2 b_2 p^4].$$

According to the conversion theorem [46] we obtain solution of the problem in the form of contour integrals:

$$\frac{\xi^0}{\chi Q} \dot{w}_0(x^0, t^0) = \sum_{k=1,2} \frac{1}{2\pi i} \int_{L_k} \frac{A_k \exp(pt^0 - n_k |x^0|)}{p n_k [(n_1^2 - n_3^2) A_1 + (n_2^2 - n_3^2) A_2]} dp - $$
$$- \frac{1}{2\pi i} \int_{L_0} \frac{(A_1 + A_2) \exp(pt^0 - n_3 |x^0|)}{p n_3 [(n_1^2 - n_3^2) A_1 + (n_2^2 - n_3^2) A_2]} dp; \tag{4.9}$$

$$\frac{\xi^0}{\chi Q} \dot{u}_0(x^0, t^0) = - \sum_{k=1,2} \int_{L_k} \frac{l_k A_k \exp(pt^0 - n_k |x^0|)}{p g_k [(n_1^2 - n_3^2) A_1 + (n_3^2 - n_2^2) A_2]} dp +$$
$$+ \frac{1}{2\pi i} \int_{L_0} \frac{l_3 (A_1 + A_2) \exp(pt^0 - n_3 |x^0|)}{p g_3 [(n_1^2 - n_3^2) A_1 + (n_3^2 - n_2^2) A_2]} dp. \tag{4.10}$$

## § 5. On the Propagation of the Elastoplastic Loading Wave in the Shell

Recently we heard of attempts to develop the dynamic theory of shells for elastoplastic deformations, which considers the possibility of large sags which fact is important for the calculation of structures and buildings [5].

Let us examine the action of moving axisymmetric load on free cylindrical and conical shells.*

Let us assume that on the cylindrical free (loose) shell with a length L at the moment t = 0 an external pressure begins to act, which is symmetric with respect to the axis of rotation and spreading on the shell surface at a certain rate v, which may be either less or

*The solution was obtained by M. P. Galin by the method of characteristics.

larger than the propagation velocity of extension and bend elastic waves in shell $k_{10} = \sqrt{\frac{E}{\rho(1-\nu^2)}}$. Let us also assume that the width of load $l \leq \infty$, i.e., the load, either is removed at a moment $t_l = \frac{l}{v}$ from the shell, or remains on it.

When $v < k_{10}$ deformations in the shell will spread at a rate $k_1$, and when $v > k_1$ the entire shell behind the front of load will be deformed, i.e., deformations will spread at a rate v. Consequently, when $v < k_{10}$ it is necessary to solve the second mixed problem, and when $v > k_1$ — the third mixed problem.

Let us note that real shells are usually reinforced on ends with sufficiently powerful ribs, preventing the end sections of the shell from turning and shifting in radial direction, i.e.,

$$\varphi = 0, \text{ when } x = 0, \quad x = L, \tag{5.1}$$

$$w = 0, \text{ when } x = 0, \quad x = L. \tag{5.2}$$

In the axial direction shell ends can have either a rigid (immobile) sealing

$$u = 0 \tag{5.3}$$

or a sliding (mobile) sealing

$$T_{xx} = 0. \tag{5.4}$$

We will reproduce briefly the course of solution of the problem for different load speeds. In the case of subsonic speed of load $v < k_{10}$ (Fig. 21). On line $x = k_{10}t$ we have zero conditions: $\varphi = w = u = 0$, $\varphi_x = \varphi_t = w_x = w_t = u_x = u_t = 0$. Reaching the opposite edge of shell $x = L$, the elastic wave $x = k_{10}t$ will be reflected from the edge. The reflected wave will travel in the opposite direction according to characteristic $dx = -k_1^{(2)}dt$, separating the traveling wave region

(1) from the reflected wave region (Fig. 21); the initial velocity reflected wave will be equal, of course, to $k_{10}$.

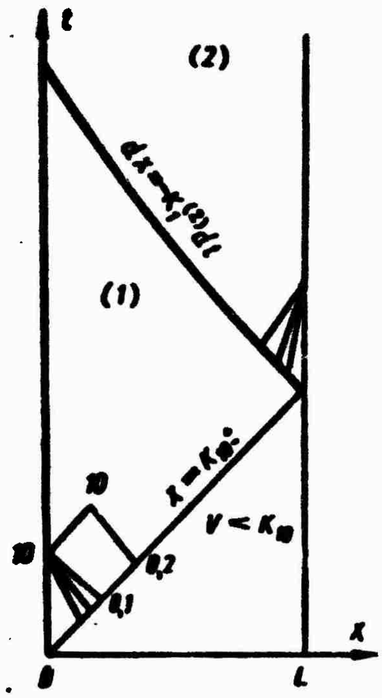


Fig. 21.

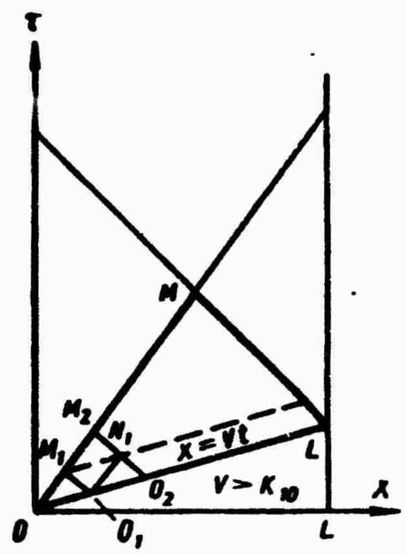


Fig. 22.

Solving the second mixed problem, we shall first determine the solution in region (1), limited by straight lines $x = k_{10}t$, $x = 0$ and by characteristic $dx = k_1^{(2)}dt$, including boundaries of this region: in a similar manner we will determine the region (2), limited by straight line $x = L$, characteristic $dx = -k_1^{(2)}dt$, on which functions u, w, φ and their first derivatives will be continuous, and by characteristic $dx = k_1^{(3)}dt$, emanating from the point of encounter of characteristic $dx = -k_1^{(2)}dt$ with straight line $x = 0$.

In the case of supersonic speed of load $v > k_{10}$ (Fig. 22). Here the solution in the region, limited by characteristics of the first family $dx = \pm k_1 dt$, emanating from point 0 (0, 0) (line OM) and point $L(L, \frac{L}{v})$ (line LM), will not depend on boundary conditions at the shell's ends, and in order to find it, we must solve the Cauchy problem according to the data on the sector of straight line $x = vt (0 \leq x \leq L, 0 \leq t \leq \frac{L}{v})$ in the triangle, limited by this straight line and

characteristics OM and LM (see Fig. 22).

From the analysis of motion equations of the cylindrical shell we see that with a load of constant intensity the solution of the Cauchy problem along straight lines, parallel to the front of motion of load $x = vt$, will not change. This circumstance significantly decreases the laboriousness of calculations in the solution of the Cauchy problem, since it will be required to determine only those points which lie directly on characteristic OM.

For the beginning of integration we will take on straight line $x = vt$ a point $O_1$, sufficiently close to point O. Now, solving the Cauchy problem according to data on the segment $OO_1$, we will define point $M_1$ on characteristic $OM_1$ of the positive direction, of the first family. In point $N_1$ lying on the intersection of characteristic of the first family of positive direction emanating from point $O_1$, with segment $M_1N_1$ of the straight line, parallel to line $x = vt$, the solution will be the same, as for point $M_1$. Therefore, the next point $M_2$ on characteristic OM, as well as all the remaining points on this characteristic can be determined either by a general method, solving the problem of Gurs according to data on characteristics $M_1O_1$ and $O_1N_1$, or, using the constancy of solution along segment $M_1N_1$, to solve Cauchy's problem according to data on segment $M_1N_1$. In both cases basic initial points i and j will be identical; however, auxiliary points, lying on characteristics of the second and third family, will be different. The preceding point (m) sought will serve as point i. In the solution of the Gurs problem the old point j will serve as point $l$. New point j will be determined from equations:

$$x_j - x_l = k_{1jl}(t_j - t_l),$$
$$x_j - x_i = v(t_j - t_i)$$

by formulas:

$$t_j = \frac{vt_i - k_{jl}t_l + x_l - x_i}{v - k_{jl}}, \quad x_j = x_i + v(t_j - t_i). \tag{5.5}$$

The remaining values, characterizing the state of the shell, such as the displacements, deformations and speeds in point j — will be the very same, as in point i.

In the determination of point m from the solution of Cauchy's problem, coordinates of point j with the given interval $\Delta x = x_j - x_i$ are determined by the formulas:

$$x_j = x_i + \Delta x, \quad t_j = t_i + \frac{\Delta x}{v}. \tag{5.6}$$

Along characteristic LM the solution is determined in the same manner, as on characteristic OM, and here the solution on characteristic LM will be equal to solution on characteristic OM, but with the time shift by $t_L = \frac{L}{v}$, i.e., for any function f we will have:

$$f_{LM}(t) = f_{OM}\left(t - \frac{L}{v}\right).$$

In the particular case of instantaneous application of uniform pressure along the entire length of the shell the solution in the corner between characteristics OM and LM and axis x in every given moment of time $t = t^*$ will be equal in all points of the segment of straight line $t = t^*$, enclosed between characteristics OM and LM.

Let us examine the problem of the same type for the conical shell. If an external load acts on the free conical shell then the component of the resultant of external forces appears, directed along the axis of symmetry of the cone which will produce a motion of the center of the cone's mass. However, until the entire shell is deformed, i.e., until perturbations reach the end x = L, it will not start to move as a rigid body, and the center of mass will be motionless.

After the perturbations reach the edge of shell $x = L$, the latter will begin to shift in the direction of the shell's axis as a rigid body, simultaneously having, of course, the displacement due to deformation. Consequently, in order to have the possibility to apply the equation to the shell in its motion with respect to the center of mass, it is necessary to stop the center of the shell's mass, and for this purpose we must apply to the center of mass d'Alembert's inertia, equal to the axial (in the considered case – horizontal) component of the external load resultant. d'Alembert's force will act on every element of the shell's mass. Therefore, acceleration of the center of mass (overload) will be equal to

$$q = \frac{P_x}{M}. \tag{5.7}$$

where $P_x$ is the horizontal component of the external load resultant: M is the mass, for instance, of the entire structure. A portion of d'Alembert's inertia will act on the shell (separately), equal to

$$P^* = qM^*. \tag{5.8}$$

Here $M^*$ is the mass of the shell. If no concentrated masses, located inside the shell are connected with the shell, then the intensity of this load can be written in the form

$$p_* = \frac{P^*}{S}, \tag{5.9}$$

where S is the surface of the shell's area. Let us note that if a tangent load will act on the cylindrical shell in addition to the external pressure, then the character of the shell's motion will be similar to the motion of the conical shell just described. Let us introduce instead of s the variable $x = s - s_0$ ($s_0$ is the distance on the generator from the summit to intersection with radius $r_0$), then for the new variable the boundary conditions with $x = 0$ and $x = L$ for the

conical shell will have the same form as for the cylindrical shell.

In solving Cauchy's problem for determination of solution on the OM characteristic (Fig. 22) we no longer will be able, in the case of a constant load, to determine all values on this characteristic, not knowing their values in the entire triangle OML, because the solution along straight lines parallel to line $x = vt$, will no longer be constant, since coefficients in motion equations will now depend on $s(r_0 = s \sin \beta)$.

However, for the conical shell of small conicity the radius of cross section $r_0$ will change little, but in length, and therefore, without any large error, for simplicity of calculation, when the length of the shell is not very great, this change can be disregarded.

We may assume that in narrow bands $\Delta x'$ wide which are adjacent to characteristics OM and LM, the solution along segments, parallel to line $x = s - s_0 = vt$, will not change even with great conicity.

Thus, in the case of the conical shell, on which a uniform load travels with the speed $v > k_{10}$, the solution on characteristics OM and LM can be found with a great degree of accuracy, without determing it in the entire triangle OML; where, in contrast to the cylindrical shell, in the actual triangle OML the solution along straight lines, parallel to OL, will change the more the larger the angle of conicity of shell and the longer the actual shell. We also note that with the small conicity of the shell, when only external pressure acts on it, it can be replaced without any great error (which essentially simplifies calculation), by a load, directed perpendicularly to the axis of the cone (without introducing d'Alembert's inertia).

Let us show an example of numerical calculation of a truncated conical steel (steel 3) shell with half-angle $\beta = 11^{\circ}$, for $\frac{r_0}{h} \approx 136$. The calculation is performed for external pressure $p_n = 112$ kg/cm$^2$ and

tangential load $p_s = 22\ \mathrm{kg/cm^2}$, spreading along the shell with the speed of 10 kg/sec in the direction of growth of $r_0$ in a band, with the width $l = 200h$, where the tangential load is directed also in the direction of growth $r_0$. In the calculation it was assumed that the material of the shell is not compressible.

The solution was found on characteristic OM. We assumed that at the distance $\Delta x = 0.85h$ in the direction of the straight line $x = vt$ the state of the shell does not change. Results of the calculation are shown in Fig. 23 and Table 3.

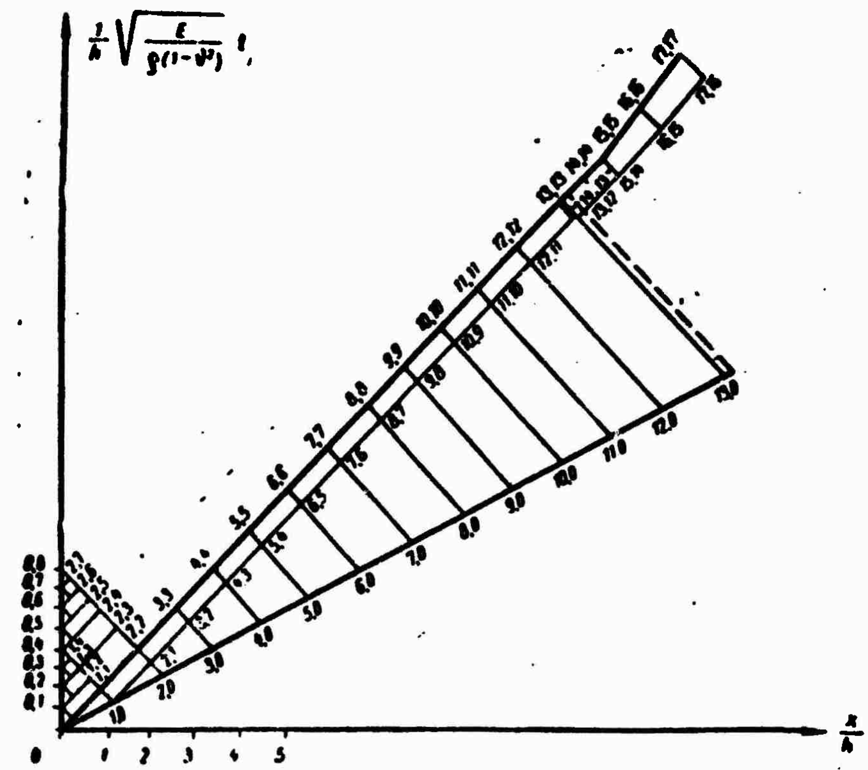


Fig. 23

It transpired that when $t = t_s = 11.1 \dfrac{h}{\sqrt{\dfrac{E}{\rho(1 - \nu^2)}}}$ in section $x = x_s = 11.1h$ plastic deformations appear for the first time in the shell opening. It was assumed that sliding stopping is used on the end $x = 0$. Results of the calculation are given in the same Fig. 23 and Table 4, as referred to above when $t_0 = 0.565 \dfrac{h}{\sqrt{\dfrac{E}{\rho(1 - \nu^2)}}}$

deformations in the stopping were still elastic.

Table 3. Change of Values Along the Characteristic of the First Family $dx = k_1 di$, Originating from Point (0, 0) and Line $\tilde{x} - x = v(\tilde{t} - t)$

| Point numbers | $\frac{x}{h}$ | $\frac{t}{h}\sqrt{\frac{E}{\rho(1-\nu^2)}}$ | $10\frac{\dot{\varepsilon}_{xx}}{\varepsilon_s}$ | $10h\frac{\varphi_s}{\varepsilon_s}$ | $10\frac{\dot{\gamma}_{xz}}{\varepsilon_s}$ | $10\frac{\varepsilon_{t1}}{\varepsilon_s}$ | $10\frac{\varepsilon_{t2}}{\varepsilon_s}$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0,0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1,1 | 0,85 | 0,85 | −0,113 | 0,359 | 0,621 | 0,312 | 0,457 |
| 2,2 | 1,70 | 1,70 | −0,231 | 0,267 | 1,90 | 1,10 | 1,17 |
| 3,3 | 2,55 | 2,55 | −0,357 | − 1,05 | 2,47 | 1,74 | 1,44 |
| 4,4 | 3,40 | 3,40 | −0,491 | − 3,05 | 3,23 | 2,94 | 2,25 |
| 5,5 | 4,25 | 4,25 | −0,629 | − 5,92 | 4,28 | 4,73 | 3,74 |
| 6,6 | 5,10 | 5,10 | −0,785 | − 9,98 | 5,74 | 7,30 | 6,03 |
| 7,7 | 5,95 | 5,95 | −0,926 | − 15,7 | 7,79 | 10,9 | 9,43 |
| 8,8 | 6,80 | 6,80 | −1,06 | − 23,9 | 10,7 | 16,0 | 14,3 |
| 9,9 | 7,65 | 7,65 | −1,19 | − 35,4 | 14,8 | 23,1 | 21,3 |
| 0,10 | 8,50 | 8,50 | −1,30 | − 51,7 | 20,6 | 33,2 | 31,2 |
| 1,11 | 9,35 | 9,35 | −1,37 | − 74,7 | 28,8 | 47,4 | 45,2 |
| 2,12 | 10,20 | 10,20 | −1,40 | −107,0 | 40,4 | 67,6 | 65,1 |
| 3,13 | 11,14 | 11,14 | −1,35 | −159,0 | 62,1 | 100,0 | 97,2 |
| 4,14 | 11,60 | 11,58 | −1,71 | −200,0 | 109 | 134,0 | 130,0 |
| 5,15 | 12,04 | 12,07 | −5,72 | −287,0 | 130,0 | 174,0 | 167,0 |
| 6,16 | 12,93 | 13,13 | −12,1 | −441,0 | 33,3 | 270,0 | 241,0 |

Table 4. (Continuation of Table 3)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 0,1 | 0 | 0,142 | −0,0000382 | −0,663 | −1,16 | 0,773 | 0,773 |
| 0,2 | 0 | 0,565 | −0,000139 | −1,08 | −1,77 | 1,20 | 1,20 |

Calculations performed show that the deformation of transverse shift in its value is approximately equal to the flexural strain (elongation-compression in external layers); now, the elongation-compression deformation of the middle surface is by a whole order smaller than flexural and shear deformation. The velocity of the elastoplastic wave of the load, appearing at point ($x_s$, $t_s$) of the OM characteristic at moment $t = t_s$ will be approximately equal to the velocity of load v. Along the LM characteristic the solution is found in a manner analogous to finding it on OM.

# CHAPTER V

## STABILITY OF SHELLS WITHIN LIMITS OF ELASTICITY

### § 1. Formulation of Problem

Let us assume that a shell is acted upon by a load, which is increasing in proportion to certain parameter $\lambda$; conditions of fastening of the shell are such that with a certain $\lambda$, for instance $\lambda = 1$, a zero moment state of strain exists.

In process of loading changes in the forms of equilibrium of shell are possible. For values of $\lambda$, smaller than a certain $\lambda_0$, there exists only a zero moment form of equilibrium of the shell, which corresponds to a minimum of energy of the "shell-external forces" system. Further, there exists a load to which corresponds number $\lambda_1 \geq \lambda_0$ which is such that when $\lambda_0 \leq \lambda \leq \lambda_1$ along with the zero-moment form of equilibrium of the shell has a moment forms also, but the zero-moment form will have a lower energy level than any moment form.

Further, we can mention number $\lambda_2 \geq \lambda_1$, which is such that although the zero-moment form of equilibrium of the shell has a relative minimum of energy when $\lambda_1 < \lambda < \lambda_2$, there is at least one moment form of equilibrium, to which corresponds a lower energy level. Finally, when $\lambda > \lambda_2$ the zero-moment form of equilibrium of the shell, in general,

ceases to be a minimum of energy point.*

Such a change of the forms of equilibrium is established in a number of investigations of the shell behavior by solving equations of nonlinear theory of shells by approximation.

In deriving [55] of such equations of the nonlinear theory of shell stability we assume that curvatures along ox and oy axes certain constant values, which phenomenon exists near second-order surfaces

$$F = \frac{1}{2}ax^2 + \frac{1}{2}by^2 + cxy + dx + ey + l = 0.$$

Consequently, the nonlinear theory under consideration is applicable to shells, the middle surface of which may be expressed by a second-order equation.

Let us direct oz axis to the normal to the middle surface in the direction of the center of curvature; we will select the origin of the coordinates in a point of the middle surface of one of the angles of the rectangular contour of the panel of the shell. Let ox and oy axes coincide with directions of the lines of the main curvatures of the shell. Let us designate the thickness of the shell with h, its dimensions along ox and oy axes with a and b (Fig. 24).

Let us assume that $k_1$ = const is the curvature of the shell, which retains the constant value along the ox axis; $k_2$ = const is the curvature, remaining constant along the oy axis. We will designate with u, v, w the displacement of points of the middle surface along ox, oy, oz axes respectively. Displacements w will characterize sags of the

---

*Expounded here is a wide-spread point of view on the stability of shells: in the author's opinion there can be other points of view. The reader will find a survey of the contemporary state of problem on the shell stability in the article by Feng Yuan Chien and Ye. Ye. Sekler "Instability of thin elastic shells." Elastic Shells, Foreign Literature, 1962. Interesting results in the USSR belong to the Kazan' school. See "Nonlinear theory of plates and shells." Publishing House of Kazan' University, 1962.

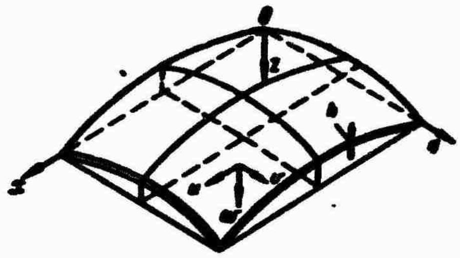


Fig. 24.

shell, the positive values of which correspond to their direction toward the center of the curvature. Sags w are not small in comparison with thickness h.

Let us assume that further $e^0_{xx}$ and $e^0_{yy}$ are linear relative deformations in the middle surface along ox and oy axes, $e^0_{xy}$ is a relative deformation of shift, $\frac{d\varkappa}{dx} = \varkappa_x$, $\frac{d\varkappa}{dy} = \varkappa_y$ are the change of curvature of a deformed shell along ox and oy axes; $\varkappa_{xy}$ is the torsion of the middle surface.

For components of deformation of the middle surface, changes of curvatures of a shell and displacements of its middle layer we obtain the following approximate relationships:

$$e^0_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - k_1 w,$$

$$e^0_{yy} = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - k_2 w, \qquad (1.1)$$

$$e^0_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y};$$

$$\varkappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \varkappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \varkappa_{xy} = -\frac{\partial^2 w}{\partial x\,\partial y}. \qquad (1.2)$$

Let us find deformations $e_{xx}$, $e_{yy}$, $e_{xy}$ for the layer, located at a distance z from the middle layer.

According to the hypothesis of straight normals we can assume that for points of this layer (Fig. 25)

$$v = v_0 - z\frac{\partial w_0}{\partial y}. \qquad (1.3)$$

Since the thickness of the shell h is small as compared to the radius of the curvature, we can consider here and subsequently that the sag of the middle layer $w_0$ is equal to the sag of any other layer of the shell.

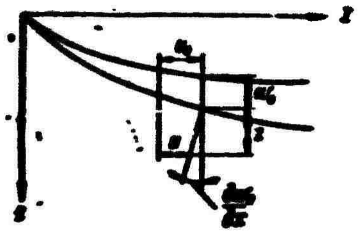


Fig. 25.

In view of that fact that sags w are considered so significant that $\left(\frac{\partial w}{\partial x}\right)^2$ and $\left(\frac{\partial w}{\partial y}\right)^2$ are values of the same order of smallness as $\frac{\partial u_0}{\partial x}$ and $\frac{\partial v_0}{\partial y}$, then (13) can be rewritten in the form

$$e_{yy}=e_{yy}^0-z\frac{\partial^2 w}{\partial y^2}.$$

and by analogy:

$$e_{xx}=e_{xx}^0-z\frac{\partial^2 w}{\partial x^2}, \qquad (1.4)$$
$$e_{xy}=e_{xy}^0-2z\frac{\partial^2 w}{\partial x\,\partial y},$$

where values $e_{xx}^0$, $e_{yy}^0$, and $e_{xy}^0$ are determined by formulas (1.1). Moreover on the strength of the hypothesis of straight normals $e_{yz} = e_{zx} = 0$. According to Hooke law, component of deformations and stresses are connected with one another by relationship:

$$e_{xx}=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)], \quad e_{xy}=\frac{2(1+\nu)}{E}\tau_{xy},$$
$$e_{yy}=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)], \quad e_{yz}=\frac{2(1+\nu)}{E}\tau_{yz}, \qquad (1.5)$$
$$e_{zz}=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)], \quad e_{zx}=\frac{2(1+\nu)}{E}\tau_{zx},$$

where $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$, and $\tau_{zx}$ are components of stresses. From these equalities it follows that:

$$\sigma_x=\frac{E}{1-\nu^2}(e_{xx}+\nu e_{yy}),$$
$$\sigma_y=\frac{E}{1-\nu^2}(e_{yy}+\nu e_{xx}), \qquad (1.6)$$
$$\tau_{xy}=\frac{E}{2(1+\nu)}e_{xy}.$$

Here $\nu$ is Poisson's ratio, E is the elastic modulus. Introducing here value $e_{xx}$, ... from (1.4), we have:

$$\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon^0_{xx} - z\,\frac{\partial^2 w}{\partial x^2} + \nu\left(\varepsilon^0_{yy} - z\,\frac{\partial^2 w}{\partial y^2}\right)\right] =$$
$$= \frac{E}{1-\nu^2}\left(\varepsilon^0_{xx} - \nu\varepsilon^0_{yy}\right) - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right).$$

.or

$$\begin{aligned}
\sigma_x &= \sigma^0_x - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right),\\
\sigma_y &= \sigma^0_y - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right),\\
\tau_{xy} &= \tau^0_{xy} - \frac{Ez}{1-\nu^2}(1-\nu)\,\frac{\partial^2 w}{\partial x\,\partial y}.
\end{aligned} \tag{1.7}$$

where it is designated that:

$$\begin{aligned}
\sigma^0_x &= \frac{E}{1-\nu^2}\left(\varepsilon^0_{xx} + \nu\varepsilon^0_{yy}\right),\\
\sigma^0_y &= \frac{E}{1-\nu^2}\left(\varepsilon^0_{yy} + \nu\varepsilon^0_{xx}\right),\\
\tau_{xy} &= \frac{E}{1-\nu^2}\,\frac{1-\nu}{2}\,\varepsilon^0_{xy}
\end{aligned} \tag{1.8}$$

are components of stresses in the middle layer.

Let us separate the element of the shell by planes, parallel to the coordinate. Forces, acting on one unit of the width of the section of the element, will be (Fig. 26):

$$T_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\,dz, \quad T_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y\,dz, \quad S_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\,dz, \quad S_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx}\,dz,$$

$$N_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx}\,dz, \quad N_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zy}\,dz, \quad M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z\,dz, \quad M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z\,dz, \tag{1.9}$$

$$H_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z\,dz, \quad H_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx} z\,dz.$$

These forces are considered positive, if their directions coincide with the positive directions of external normals towards +ox

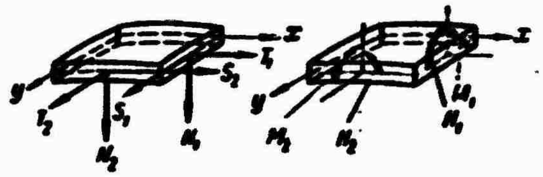


Fig. 26.

and +oy. By virtue of relationship

$$\tau_{xy} = \tau_{yx},$$

known from theory of elasticity, one may assume that

$$S_1 = S_2 = S, \qquad H_1 = H_2 = H. \tag{1.10}$$

Let us introduce in (19) value $\sigma_x$, $\sigma_y$, $\tau_{xy}$ from (1.7). We will obtain for $M_1$:

$$M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z\,dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\left[\sigma_x^0 - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right]dz =$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x^0 z\,dz - \frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\,dz = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$ is cylindrical rigidity. Analogously we will find $M_2$ and N. Consequently:

$$\begin{aligned} M_1 &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \\ M_2 &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \\ H &= -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}. \end{aligned} \tag{1.11}$$

Substituting now in expressions (1.9) for $T_1$, ..., $N_1$, $N_2$ values $\sigma_x$, $\sigma_y$, ..., $\tau_{xy}$ from (1.7), taking into account (1.8) and (1.1), we have:

$$T_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\,dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_x^0 - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right]dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{E}{1-\nu^2}(e_{xx}^0 + \nu e_{yy}^0) -\right.$$

$$\left. - \frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right]dz = \frac{E}{1-\nu^2}[e_{xx}^0 + \nu e_{yy}^0]\left(\frac{h}{2} + \frac{h}{2}\right) - \frac{E}{2(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{h^2}{4} - \frac{h^2}{4}\right) =$$

$$= \frac{Eh}{1-\nu^2}(e_{xx}^0 + \nu e_{yy}^0).$$

Introducing here values $e_{xx}^{0}$ and $e_{yy}^{0}$ from (1.1), we will obtain:

$$T_1=\frac{Eh}{1-\nu^2}(e_{xx}^0+\nu e_{yy}^0)=\frac{Eh}{1-\nu^2}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-k_1w+\right.$$
$$\left.+\nu\frac{\partial v_0}{\partial y}+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2-\nu k_2w\right]=\frac{Eh}{1-\nu^2}\left[\frac{\partial u_0}{\partial x}+\nu\frac{\partial v_0}{\partial y}+\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2-k_1w-\nu k_2w\right].$$

Analogously we will find $T_2$ and $S = S_1 = S_2$.

Thus we obtain important formulas:

$$T_1=\frac{Eh}{1-\nu^2}\left[\frac{\partial u_0}{\partial x}+\nu\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2-k_1w-\nu k_2w\right],$$
$$T_2=\frac{Eh}{1-\nu^2}\left[\frac{\partial v_0}{\partial y}+\nu\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2-\right.$$
$$\left.-k_2w-\nu k_1w\right], \qquad (1.12)$$
$$S=\frac{Eh}{2(1+\nu)}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right),$$

which produce a relationship between tangential forces, effective in the middle layer, and its displacements. Shearing forces will be determined below.

Excluding unknown displacements of the middle layer $u_0$ and $v_0$ from equalities (1.12), we can express forces $T_1$, $T_2$, S through sags w.

First let us note that from (1.12) we can obtain the following relationships:

$$T_1-\nu T_2=Eh\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-k_1w\right],$$
$$T_2-\nu T_1=Eh\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-k_2w\right], \qquad (1.13)$$
$$S=\frac{Eh}{2(1+\nu)}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right).$$

Using these equalities, let us set up equation

$$\frac{\partial^2}{\partial y^2}(T_1 - \nu T_2) + \frac{\partial^2}{\partial x^2}(T_2 - \nu T_1) - 2(1+\nu)\frac{\partial^2 S}{\partial x \partial y} =$$

$$= Eh\left\{\frac{\partial^2}{\partial y^2}\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - k_1 w\right] + \right.$$

$$+ \frac{\partial^2}{\partial x^2}\left[\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - k_2 w\right] - \left.\frac{\partial^2}{\partial x \partial y}\left[\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\} =$$

$$= Eh\left\{\frac{\partial^2}{\partial y^2}\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) - \right.$$

$$\left. - k_1\frac{\partial^2 w}{\partial y^2} - k_2\frac{\partial^2 w}{\partial x^2}\right\}.$$

But, according to rules of differentiation

$$\frac{\partial^2}{\partial y^2}\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = \frac{\partial}{\partial y}\left[\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right] = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2},$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] = \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right] = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2 \partial y},$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) =$$

$$= \frac{\partial^3 w}{\partial x^2 \partial y}\frac{\partial w}{\partial y} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2}.$$

Introducing this into the preceding equation, we will obtain:

$$\frac{\partial^2}{\partial y^2}(T_1 - \nu T_2) + \frac{\partial^2}{\partial x^2}(T_2 - \nu T_1) - 2(1+\nu)\frac{\partial^2 S}{\partial x \partial y} =$$

$$= Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - k_1\frac{\partial^2 w}{\partial y^2} - k_2\frac{\partial^2 w}{\partial x^2}\right]. \tag{1.14}$$

Equation (1.14) gives us the sought relationship between forces, effective in the middle plane of the shell and its sag w.

Introducing the stress function $\varphi$ by formulas:

$$T_1 = h\frac{\partial^2 \varphi}{\partial y^2}, \quad T_2 = h\frac{\partial^2 \varphi}{\partial x^2}, \quad S = -h\frac{\partial^2 \varphi}{\partial x \partial y}, \tag{1.15}$$

equation (1.14), after substitution of values (1.15), will assume the form:

$$\frac{1}{E}\nabla^2\nabla^2\varphi + k_1\frac{\partial^2 w}{\partial y^2} + k_2\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = 0, \tag{1.16}$$

where

$$\nabla^2\nabla^2\varphi = \frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4}. \tag{1.17}$$

$\nabla^2\nabla^2(\quad)$ is a biharmonic Laplacian operator.

Now let us turn to the construction of the equation of equilibrium.

Let us assume that on the shell act the following load: transverse $q = q(x, y)$, effective normally to middle surface; compressing or stretching forces $hp(x)$ and $hr(y)$, applied normally to the edges of the shell; and shear forces $h\tau$, acting along the edges.

Let us separate element $hdxdy$ from the shell by two pairs of mutually perpendicular planes, parallel prior to the deformation of the shell, to planes $xoz$ and $yoz$.

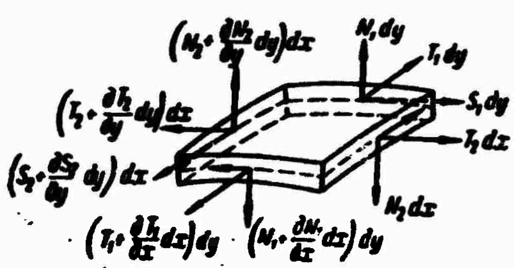



Fig. 27.

Forces, acting on the edges of this element, are shown in Fig. 27.

When the middle layer is bent these forces turn in space. Projecting them on mobile coordinate axes and rejecting small values of the higher order, we will obtain:

$$\left(T_1 + \frac{\partial T_1}{\partial x}dx\right)dy - T_1\,dy + \left(S_2 + \frac{\partial S_2}{\partial y}dy\right)dx - S_2\,dx -$$
$$- N_1\,dy\frac{1}{2}\frac{\partial^2 w}{\partial x^2}dx - \left(N_1 + \frac{\partial N_1}{\partial x}dx\right)dy\frac{1}{2}\frac{\partial^2 w}{\partial x^2}dx = 0,$$

$$- T_2\,dx + \left(T_2 + \frac{\partial T_2}{\partial y}dy\right)dx + \left(S_1 + \frac{\partial S_1}{\partial x}dx\right)dy - S_1\,dy -$$
$$- \left(N_2 + \frac{\partial N_2}{\partial y}dy\right)dx\frac{1}{2}\frac{\partial^2 w}{\partial y^2}dy - N_2\,dx\frac{1}{2}\frac{\partial^2 w}{\partial y^2}dy = 0,$$

$$\left(T_1 + \frac{\partial T_1}{\partial x}dx + T_1\right)dy\left(\frac{dx}{2R_1} + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}dx\right) +$$
$$+ \left(T_2 + \frac{\partial T_2}{\partial y}dy + T_2\right)dx\left(\frac{dy}{2R_2} + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}dy\right) +$$
$$+ \left(S_1 + \frac{\partial S_1}{\partial x}dx + S_1\right)dy\frac{1}{2}\frac{\partial^2 w}{\partial x\,\partial y} +$$
$$+ \left(S_2 + \frac{\partial S_2}{\partial y}dy + S_2\right)dx\frac{1}{2}\frac{\partial^2 w}{\partial x\,\partial y}dy + \left(N_1 + \frac{\partial N_1}{\partial x}dx - N_1\right)dy +$$
$$+ \left(N_2 + \frac{\partial N_2}{\partial y}dy - N_2\right)dx + q(x, y)\,dx\,dy = 0.$$

Performing reductions in the first two equations and rejecting terms of the third order of smallness, we have:

$$\frac{\partial T_1}{\partial x}+\frac{\partial S_2}{\partial y}=N_1\frac{\partial^2 w}{\partial x^2}.$$
$$\frac{\partial T_2}{\partial y}+\frac{\partial S_1}{\partial x}=N_2\frac{\partial^2 w}{\partial y^2}. \tag{1.18}$$

The third equation after opening of brackets will be

$$\begin{aligned}&2T_1\frac{dx}{2R_1}dy+2T_1\frac{1}{2}\frac{\partial^2 w}{\partial x^2}dx\,dy+\frac{\partial T_1}{\partial x}dx\,dy\frac{dx}{2R_1}+\\&+\frac{\partial T_1}{\partial x}dx\frac{1}{2}\frac{\partial^2 w}{\partial x^2}dx\,dy+2T_2dx\frac{dy}{2R_2}+2T_2dx\frac{1}{2}\frac{\partial^2 w}{\partial y^2}dy+\\&+\frac{\partial T_2}{\partial y}dy\frac{1}{2}\frac{\partial^2 w}{\partial y^2}dx\,dy+2S_1\,dy\frac{1}{2}\frac{\partial^2 w}{\partial x\,\partial y}dx+\\&+\frac{\partial S_1}{\partial x}dx\,dy\frac{1}{2}\frac{\partial^2 w}{\partial x\,\partial y}dx+\frac{\partial S_1}{\partial x}dx\,dy\frac{dx}{2R_1}+2S_2dx\frac{1}{2}\frac{\partial^2 w}{\partial x\,\partial y}dy+\\&+\frac{\partial S_2}{\partial y}dy\,dx\frac{dy}{2R_2}+\frac{\partial S_2}{\partial y}dy\,dx\frac{1}{2}\frac{\partial^2 w}{\partial x\,\partial y}dy+\frac{\partial N_1}{\partial x}dx\,dy+\\&+\frac{\partial N_2}{\partial y}dy\,dx+q\,dx\,dy=0.\end{aligned} \tag{1.19}$$

Taking into consideration relationship (1.10) and rejecting values of this third order of smallness, after reductions we will obtain:

$$\begin{aligned}&T_1\frac{1}{R_1}+T_1\frac{\partial^2 w}{\partial x^2}+T_2\frac{1}{R_2}+T_2\frac{\partial^2 w}{\partial y^2}+S_1\frac{\partial^2 w}{\partial x\,\partial y}+S_2\frac{\partial^2 w}{\partial x\,\partial y}+\\&+\frac{\partial N_1}{\partial x}+\frac{\partial N_2}{\partial y}+q=0,\end{aligned}$$

or

$$\begin{aligned}\frac{\partial N_1}{\partial x}+\frac{\partial N_2}{\partial y}=&-q-\frac{T_1}{R_1}-\frac{T_2}{R_2}-T_1\frac{\partial^2 w}{\partial x^2}-T_2\frac{\partial^2 w}{\partial y^2}-\\&-2S\frac{\partial^2 w}{\partial x\,\partial y}.\end{aligned} \tag{1.20}$$

Comparing equation of moments of these forces with respect to the ox axis (Fig. 28), we have:

$$\begin{aligned}&\left(M_1+\frac{\partial M_1}{\partial x}dx\right)dy-M_1dy-H_2dx+\left(H_2+\frac{\partial H_2}{\partial y}dy\right)dx-\\&-\left(N_1+\frac{\partial N_1}{\partial x}dx+N_2\right)dy\frac{1}{2}dx=0.\end{aligned} \tag{1.21}$$

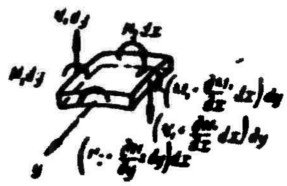


Fig. 28.

which, after simplification and calculation (1.11) yields

$$N_1 = \frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial y} = -D\frac{\partial}{\partial x}\nabla^2 w. \qquad (1.22)$$

Analogously we will obtain:

$$N_2 = \frac{\partial M_2}{\partial y} + \frac{\partial H}{\partial x} = -D\frac{\partial}{\partial y}\nabla^2 w.$$

Relationships (1.22) express equality to zero of the main moment of all forces, acting on the shell element examined. With the help of equalities (1.22) we can exclude from (1.21) the severing forces $N_1$ and $N_2$. For this we will introduce $N_1$ and $N_2$ from (1.21) into (1.22). We have:

$$\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 M_2}{\partial y^2} + T_1\left(k_1 + \frac{\partial^2 w}{\partial x^2}\right) + T_2\left(k_2 + \frac{\partial^2 w}{\partial y^2}\right) + 2S\frac{\partial^2 w}{\partial x\,\partial y} + q = 0. \qquad (1.23)$$

Equations (1.18) and (1.23) give us the sought totality of fundamental equations of equilibrium. The right sides of equations (1.18) can be assumed to be equal to zero, since values $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$ have the order of $\frac{w}{b^2}$, and $N_1 \sim \frac{h^3 w}{b^3}$, $N_2 \sim \frac{h^3 w}{a^3}$, then the right sides of equations (1.18) have the order of $\frac{h^3 w^2}{b^5}$, $\frac{h^3 w^2}{a^5}$, whereas $\frac{\partial T_1}{\partial x}$, $\frac{\partial S}{\partial y}$ and others will have the order $\frac{hw^2}{b^3}$, $\frac{hw^2}{a^3}$.

Introducing according to formulas (1.15) the function of stresses, we can easily see that equations (1.18) will be identically satisfied. Taking into consideration expression (1.11) for moments, after introduction of function of stresses, equation (1.23) will be rewritten thus:

$$-D\left(\frac{\partial^4 w}{\partial x^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-2D(1-\nu)\frac{\partial^4 w}{\partial x^2\partial y^2}-D\left(\frac{\partial^4 w}{\partial y^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)+$$
$$+h\frac{\partial^2\varphi}{\partial y^2}\left(k_1+\frac{\partial^2 w}{\partial x^2}\right)+h\frac{\partial^2\varphi}{\partial x^2}\left(k_2+\frac{\partial^2 w}{\partial y^2}\right)-2\frac{\partial^2\varphi}{\partial x\,\partial y}h\frac{\partial^2 w}{\partial x\,\partial y}+q=0.$$

or

$$-D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+$$
$$+h\left(k_1\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+k_2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)+q=0.$$

will be written in an abbreviated form thus:

$$D\nabla^2\nabla^2 w-h\left(k_1\frac{\partial^2\varphi}{\partial y^2}+k_2\frac{\partial^2\varphi}{\partial x^2}\right)-$$
$$-h\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)-q=0, \tag{1.24}$$

where $\nabla^2\nabla^2(\ \ )$ is a biharmonic Laplacian operator.

The equation of equilibrium (1.24) is the second fundamental equation of the theory of flexible sloping shells.

We must note that in setting up fundamental equations (1.16) and (1.24) ox and oy coordinate axes were assumed to coincide with the main curvatures. In the more general case, when there is no such coincidence, equations (1.16) and (1.24) will contain component

$$k_{12}=\frac{\partial^2 F}{\partial x\,\partial y},$$

where F = F(x, y) is the equation of the middle surface of the shell. This term takes into account the influence of the curvature of torsion on deformation and stressed state of the shell. Assuming that in (1.16) and (1.24) the initial curvatures are $k_1 = 0$ and $k_2 = 0$, we will obtain equations for plates with a large sag:

$$\nabla^2\nabla^2\varphi=E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right],$$
$$D\nabla^2\nabla^2 w-q=h\left[\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right].$$

The problem on the bend and stability of plates and shells, as we can see from the preceding facts, is reduced to integration of the system of consistent nonlinear partial differential equations of the fourth order:

$$\frac{1}{E}\nabla^2\nabla^2\varphi+k_1\frac{\partial^2 w}{\partial y^2}+k_2\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2=0; \tag{1.16}$$

$$D\nabla^2\nabla^2 w-h\left(k_2\frac{\partial^2\varphi}{\partial x^2}+k_1\frac{\partial^2\varphi}{\partial y^2}\right)-$$

$$-h\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)-q=0. \tag{1.25}$$

The relationship between forces, acting in the middle layer, and the displacements is determined by formulas (1.12):

$$T_1=\frac{Eh}{1-\nu^2}\left[\frac{\partial u_0}{\partial x}+\nu\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2-k_1 w-\nu k_2 w\right],$$

$$T_2=\frac{Eh}{1-\nu^2}\left[\frac{\partial v_0}{\partial y}+\nu\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2-k_2 w-\nu k_1 w\right], \tag{1.26}$$

$$S=\frac{Eh}{2(1+\nu)}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right).$$

In the case of the spherical shell ($k_1 = k_2 = k$) equations (1.16) and (1.25) will assume the form:

$$\frac{1}{E}\nabla^2\nabla^2\varphi+k\nabla^2 w+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2=0,$$

$$D\nabla^2\nabla^2 w-hk\nabla^2\varphi-h\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)-q=0.$$

For the cylindrical shell (for instance, $k_1 = 0$, $k_2 \neq 0$) the equations will be as follows:

$$\frac{1}{E}\nabla^2\nabla^2\varphi+k_2\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2=0,$$

$$D\nabla^2\nabla^2 w-hk_2\frac{\partial^2\varphi}{\partial x^2}-h\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)-q=0.$$

If signs of curvatures $k_1$ and $k_2$ are different, for instance $k_1 > 0$, $k_2 < 0$, then these equations can be used as fundamental

equations of the theory of flexible shells of negative Gaussian curvature.

Dropping the nonlinear terms in equations (1.16) and (1.25) we will obtain fundamental equations of shells with small sag:

$$\frac{1}{E}\nabla^2\nabla^2\varphi + k_1\frac{\partial^2 w}{\partial y^2} + k_2\frac{\partial^2 w}{\partial x^2} = 0,$$

$$D\nabla^2\nabla^2 w - h\left(k_2\frac{\partial^2\varphi}{\partial x^2} + k_1\frac{\partial^2\varphi}{\partial y^2}\right) - q = 0,$$

to which the linear technical theory of shells is reduced.

If in the classical theory bends of plates are usually prescribed by two boundary conditions with respect to w on edges, then in the nonlinear theory these conditions are already insufficient. In addition to two conditions about sags here we must prescribe two more conditions on every edge with respect to the function φ.

Instead of boundary conditions relative to φ we can prescribe displacements u and v, which are connected with w and φ by conditions (1.26). In the nonlinear theory of bending of plates concepts of hinged support, rigid clamping, and others require a somewhat more precise definition. For instance, diagrams of fastening of edges (Fig. 29) in the linear theory correspond to the idea of hinged fastening of edges. However, working conditions of these plates with large sags will be different.

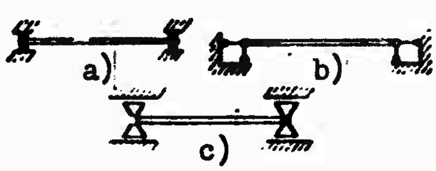



Fig. 29.

Boundary conditions of diagram (a) can be recorded thus. Sag w and moment on edges x = const turn into zero:

$$w = 0, \quad M_1 = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0,$$

where, by virtue of the first of these boundary conditions, the second one will turn into a simpler

$$\frac{\partial^2 w}{\partial x^2} = 0.$$

Furthermore, since no external forces are applied on the contour then normal and tangent forces on the edges are equal to zero:

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad \tau = -\frac{\partial^2 \varphi}{\partial x \partial y} = 0.$$

In diagram (b) nothing can be said beforehand about forces on edges, but about displacements one may assume that they turn into zero on the edges.

Fig. 30.

Consequently, boundary conditions for fastening of edges according to diagram (b) will be written thus:

$$w = \frac{\partial^2 w}{\partial x^2} = 0, \quad u = v = 0.$$

These conditions indicate the fact that the geometry of the shell's edges when it is bent remains the same and they are not displaced.

In the third case (c) of fastening of edges the boundary conditions will be:

$$w = \frac{\partial^2 w}{\partial x^2} = 0,$$
$$u = \text{const}, \quad v = \text{const},$$

which expresses the fact of displacement of the panel's edges during deformation of the shell parallel to themselves.

Combinations of these methods of hinged fastening are also possible.

We can reason analogously also in the case of clamped edges (Fig. 30).

Boundary conditions, corresponding to the diagram (c), can be written thus:

$$w = \frac{\partial w}{\partial x} = 0,$$
$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad \tau = -\frac{\partial^2 \varphi}{\partial x \partial y} = 0,$$

and boundary conditions of diagram (d) will be

$$w = \frac{\partial w}{\partial x} = u = v = 0.$$

In the first case (c) upon deformation of plates the edges are warped. In the second (d) the edges remain rectilinear and are not displaced.

In the linear theory there is no necessity for such distinction and diagrams (c) and (d) can be considered equivalent.

In the case, when on plate edges displacements u and v are prescribed after integration of fundamental equations (1.16) and (1.25) according to the function found $\varphi$ and w we should set up general expressions of displacements u and v, which we should then subordinate to the prescribed boundary conditions on the edges. Other conditions are not considered here.

## § 2. Panel of a Flexible Sloping Shell

Let us examine problems on the bend and stability of panels of flexible sloping shells (geometric nonlinearity).

The term flexible is applied to such shells, for which sag is comparable with their thickness. For that we take into consideration geometric nonlinearity, expressing the deformation through displacements in the middle surface by relationships (1.1) of this chapter, i.e., in the series for components of deformation we retain terms of the second order of smallness with respect to sags w(x, y). The problem may be reduced to integration of a system of two nonlinear equations (1.16) and (1.25):

$$\Phi \equiv \frac{1}{E}\nabla^2\nabla^2\varphi + k_1\frac{\partial^2 w}{\partial y^2} + k_2\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 = 0, \qquad (2.1)$$

$$\begin{gathered} \Psi \equiv D\nabla^2\nabla^2 w - h\left(k_2\frac{\partial^2\varphi}{\partial x^2} + k_1\frac{\partial^2\varphi}{\partial y^2}\right) - \\ -h\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right) - q = 0, \end{gathered} \qquad (2.2)$$

under corresponding boundary conditions.

The majority of problems of the nonlinear theory of elastic shells is solved by approximation methods, while, in view of their complexity, we usually limit ourselves to a solution in series in the first approximation. We can acquaint ourselves with many solutions in books by A. S. Vol'mir[4] and Kh. M. Mushtari and K. Z. Galimov [47]. Usually we apply Ritz and Bubnov-Galerkin methods.

Let us give here the solution of problems on the bend and stability of a flexible sloping shell, constructed by M. A. Koltunov in the first [48] and higher approximations [49] by the Bubnov-Galerkin method.

Solutions of such problems present an interest from the point of view of possibility of establishment of a region of instability of

shells. The linear theory of shells enables us to establish only the upper boundary of this region. Experiments show that values of calculated critical stresses do not coincide with experimental values and exceed them significantly.

Examination of the problem on the bend and stability of shells in the light of the nonlinear theory enables us to foresee the behavior of the shell after loss of stability and to establish not only the upper boundary of the region of instability, corresponding to critical stresses, obtained by the linear theory, but its lower bound also.

Let us present certain considerations of a general character about the solution of problems of this type. Let us assume that to the shell, which is somehow secured on hard piecewise smooth frame, we apply an arbitrary transverse load q(x, y) and compressing or stretching stresses normal to edges, the components of which along oy and ox axes will be p(x) and r(y). It is required to determine the relationship between external forces acting on the shell and its sags, not considering the latter to be small. For the solution of this problem it is necessary to integrate equations (2.1) and (2.2). In view of the fact that methods of exact integration of these equations so far have not been found, we will look for their approximate solutions in the form of series:

$$\varphi = \sum_m \sum_n A_{mn} [U_m(x) V_n(y) - \theta(x) - \lambda(y)],$$

$$w = \sum_m \sum_n f_{mn} X_m(x) Y_n(y),$$

where $A_{mn}$ and $f_{mn}$ are unknown constants, but functions $U_m(x)$, $V_n(y)$, $X_m(x)$, $Y_n(y)$ are selected beforehand in such a way, as to satisfy all static and geometric contour conditions.

Applying the Bubnov-Galerkin method, we will enter these expressions for $\varphi$ and w in equations (2.1) and (2.2), then, multiplying in accordance with their physical sense the first one by the variation of function of $\varphi$, and the second one by the variation of function of w and taking into consideration the independence of variations of parameters $\delta A_{mn}$ among themselves and variations of parameters $\delta f_{mn}$ among themselves, we will integrate the expressions obtained with respect to region, limited by the contour of the shell. From the system of nonlinear algebraic equations obtained we will find unknown parameters $A_{mn}$ and $f_{mn}$.

The feasibility of application of the Bubnov-Galerkin method to operators $\Phi$ and W was studied in the works of I. I. Vorovich [50] and other authors. The same subject is related in [51], where the proof, proposed by A. R. Rzhanitskiy is adduced.

According to the physical sense, continuity equations of deformations (2.1) and equilibrium (2.2) must tolerate one solution each and be the conditions of extremum of a certain function $Q(w, \varphi)$. Let us assume that such a functional exists. Necessary conditions of extremum of functional $Q(w, \varphi)$ with respect to w and $\varphi$:

$$\delta_w Q(w, \varphi) = 0, \quad \delta_\varphi Q(w, \varphi) = 0,$$

must coincide with equations (2.2) and (2.1) i.e.,

$$\delta_\varphi Q(w, \varphi) = \Phi(w, \varphi), \quad \delta_w Q(w, \varphi) = W(w, \varphi).$$

Modifying these equalities, we will obtain:

$$\delta_w [\delta_\varphi Q(w, \varphi)] = \delta_w \Phi(w, \varphi), \quad \delta_\varphi [\delta_w Q(w, \varphi)] = \delta_\varphi W(w, \varphi).$$

Subtracting one equality from the other, we will obtain the necessary condition with which equations $\Phi = 0$ and $W = 0$ are conditions of extremum of functional Q.

$$\delta_w \Phi(w, \varphi) = \delta_\varphi \Psi(w, \varphi).$$

It is not difficult to verify that for equations (2.1) and (2.2) this equality is fulfilled.

Indeed, Euler's equation gives for the first one of them:

$$\delta_w \Phi = \frac{\partial \Phi}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Phi}{\partial w_{xx}}\right) +$$
$$+ 2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial \Phi}{\partial w_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial \Phi}{\partial w_{yy}}\right) + \ldots = E\left[\frac{\partial^2}{\partial x^2}\left(-\frac{\partial^2 w}{\partial y^2}\right) +\right.$$
$$\left. + 2\frac{\partial^2}{\partial x\,\partial y}\left(2\frac{\partial^2 w}{\partial x\,\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(-\frac{\partial^2 w}{\partial x^2}\right)\right] = 2E\frac{\partial^4 w}{\partial x^2\,\partial y^2}.$$

Analogously we will obtain:

$$\delta_\varphi \Psi = \frac{h}{D}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\left(-2\frac{\partial^2 w}{\partial x\,\partial y}\right) +\right.$$
$$\left. + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}\right)\right] = 2\frac{h}{D}\frac{\partial^4 w}{\partial x^2\,\partial y^2}.$$

Consequently, this equality is fulfilled with an accuracy up to a constant factor. Multiplying (2.1) by $\frac{h}{ED}$, we will obtain the exact fulfillment of the condition of applicability of the Bubnov-Galerkin method to the solution of nonlinear problems of elastic shells.

Let us examine the solution of the problem in the first approximation. We will assume that functions U(x), ..., Y(y) are selected so that all boundary conditions are satisfied. We will write the solution in the following form:

$$\varphi = A[U(x)V(y) - \theta(x) - \lambda(y)], \tag{2.3}$$
$$w = fX(x)Y(y). \tag{2.4}$$

where functions $\theta(x)$ and $\lambda(y)$ are selected so that

$$\theta''(x) = \frac{1}{A}p(x), \quad \lambda''(y) = \frac{1}{A}r(y). \tag{2.5}$$

Setting up the Bubnob-Galerkin equation:

$$\iint_{(a)} \Phi(U, V, X, Y, A, f, p, r)\,UV\,dx\,dy = 0,$$

$$\iint_{(a)} \Psi(U, V, X, Y, A, f, p, r, q)\,XY\,dx\,dy = 0,$$

where (G) is the region, limited by the contour of the shell, and integrating, we will obtain:

$$\frac{A}{E} I_1 - \frac{A}{E} I_2 + f I_3 + f^2 I_4 = 0. \tag{2.6}$$

$$-AhI_5 + AhI_6 + DfI_7 - AfhI_8 + AfhI_9 - I_{10} = 0. \tag{2.7}$$

Here $I_i$ are constants, depending on the dimensions of the shell, its curvature, external forces, and boundary conditions, and are determined by the following formulas:

$$\begin{aligned}
I_1 &= \iint\limits_{(G)} (U^{IV}V + 2U''V'' + UV^{IV})\, UV\, dx\, dy, \\
I_2 &= \iint\limits_{(G)} (\theta^{IV} + \lambda^{IV})\, UV\, dx\, dy, \\
I_3 &= \iint\limits_{(G)} (k_2 X''Y + k_1 XY'')\, UV\, dx\, dy, \\
I_4 &= \iint\limits_{(G)} (X''YXY'' - X'^2 Y'^2)\, UV\, dx\, dy, \\
I_5 &= \iint\limits_{(G)} (k_2 U''V + k_1 UV'')\, XY\, dx\, dy, \\
I_6 &= \iint\limits_{(G)} (k_2 \theta'' + k_1 \lambda'')\, XY\, dx\, dy, \\
I_7 &= \iint\limits_{(G)} (X^{IV}Y + 2X''Y'' + XY^{IV})\, XY\, dx\, dy, \\
I_8 &= \iint\limits_{(G)} (UV''X''Y + U''VXY'' - 2U'V'X'Y')\, XY\, dx\, dy, \\
I_9 &= \iint\limits_{(G)} (\lambda''X''Y + \theta''XY'')\, XY\, dx\, dy, \\
I_{10} &= \iint\limits_{(G)} q(x, y)\, XY\, dx\, dy.
\end{aligned} \tag{2.8}$$

Calculating these integrals for either form of fastening of edges and prescribed loads and then introducing them in equations (2.6) and (2.7), we will obtain a solution of the problem posed. Determining from (2.6) the value

$$A = -\frac{EfI_3 + Ef^2I_4}{I_1 - I_2} \tag{2.9}$$

and introducing it in equation (2.7), we will obtain:

$$I_{10} = DfI_7 - \frac{EhfI_3 + Ehf^3I_4}{I_1 - I_2}(I_6 - I_5 - fI_8 + fI_9). \quad (2.10)$$

Equality (2.10) yields the sought relationship between the load and the sag in the center of the shell. Subsequently we will term equality (2.10) the general solution in the first approximation of nonlinear problem of bend of sloping shells under any conditions of fastening of their edges on piecewise-smooth contour and under any loads, prescribed on its edges and acting in the normal sense toward its middle surface.

For the case of a plane plate the equation (2.10) will assume the form

$$I_{10} = DfI_7 - \frac{I_4(I_9 - I_8)}{I_1 - I_2} Ehf^3, \quad (2.11)$$

which gives us the solution for compressed-bent plates of finite rigidity.

Let us adduce the solution of the problem in the case of hinged fastening of edges of the shell panel, having in its plan the shape of a quadrangle.

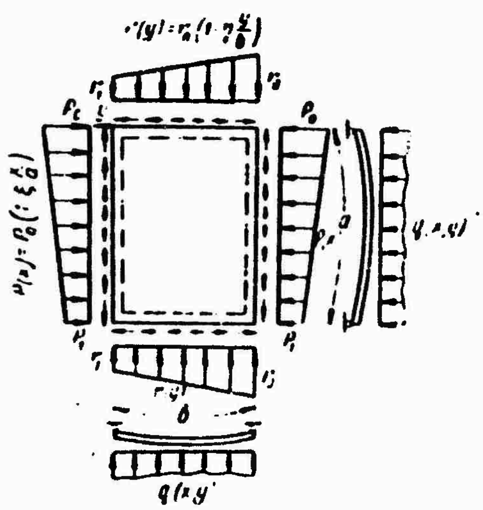


Fig. 31.

Let us assume that the shell is subjected to the action of an arbitrary lateral load q(x, y) and stresses applied to the edges of stresses, distributed along the edges according to the linear law (Fig. 31). The solution should satisfy the following boundary conditions:

$$\begin{aligned} w &= \frac{\partial^2 w}{\partial x^2} = 0, \quad \text{when } x = 0, \ x = a; \\ w &= \frac{\partial^2 w}{\partial y^2} = 0, \quad \text{when } y = 0, \ y = b; \end{aligned} \quad (2.12)$$

$$\sigma_x^\circ = \frac{\partial^2\varphi}{\partial y^2} = r(y) = r_0\left(1 - \eta\frac{y}{b}\right),$$
$$\tau^\circ = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \text{when } x = 0,\ x = a;$$
$$\sigma_y^\circ = \frac{\partial^2\varphi}{\partial x^2} = p(x) = p_0\left(1 - \xi\frac{x}{a}\right),$$
$$\tau^\circ = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \text{when } y = 0,\ y = b. \qquad \text{(2.12 continued)}$$

where

$$\xi = 1 - \frac{p_0}{p_1}, \quad \eta = 1 - \frac{r_0}{r_1}.$$

Here we examine a shell with Gaussian curvature $\Gamma = k_1 k_2$, which is positive, negative, or equal to zero.

As approximating factors we will select fundamental beam functions:

$$U(x) = \sin\frac{\pi x}{a}, \quad V(y) = \sin\frac{\pi y}{b},$$
$$X(x) = \sin\frac{\pi x}{a}, \quad Y(y) = \sin\frac{\pi y}{b}, \qquad (2.13)$$

corresponding to the fundamental tone of oscillations of a beam hinge-secured on its ends.

Thus, solutions (2.3) and (2.4) will be written in this manner:

$$\varphi = A\left[\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - \theta(x) - \lambda(y)\right], \qquad (2.14)$$

$$w = f\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \qquad (2.15)$$

Here

$$\theta''(x) = \frac{p_0}{A}\left(1 - \xi\frac{x}{a}\right), \quad \lambda''(y) = \frac{r_0}{A}\left(1 - \eta\frac{y}{b}\right). \qquad (2.16)$$

Calculations, performed by S. P. Timoshenko [52], showed that in plate the stresses from compression predominate over stresses from bending, i.e., if coefficients $\xi$ and $\eta$ do not exceed the value of 2/3, then the expression (2.15) reflects with sufficient accuracy the bent surface of the plate. Therefore, let us construct our solution

applicably to values $\xi \le \frac{2}{3}$, $\eta \le \frac{2}{3}$. For $\xi$ and $\eta$, exceeding these values, we must use a large number of terms of the approximating series.

It is not difficult to be reassured by direct verification that in the selection of functions $\varphi$ and w in formula (2.14) and (2.15) all boundary conditions (2.12) are fulfilled exactly, except for the value of tangent forces, acting on the edges, which turns into zero on the "average," i.e.,

$$\tau_{cp} = -\frac{1}{a}\int_0^a \frac{\partial^2\varphi}{\partial x\,\partial y}\,dx = -\frac{A\pi^2}{a^2 b}\cos\frac{\pi y}{b}\int_0^a \cos\frac{\pi x}{a}\,dx = 0 \tag{2.17}$$

[cp = av = average]

on edges y = 0, y = b. We will have the same on edges x = 0 and x = a. Consequently, in the problem we assumed the presence of tangent forces on the edges of the shell, which fact has practical significance.

The diagram of loading of edges by forces $\tau$, p(x) r(y), and q(x, y) is presented in Fig. 31.

Determining derivatives (2.13) and introducing them in integrands (2.8), after integration and calculations we will obtain the following values of integrals $I_i$:

$$\begin{gathered}
I_1 = \frac{\pi^4}{4}\left(\frac{b}{a}+\frac{a}{b}\right)^2\frac{1}{ab},\\
I_2 = 0,\\
I_3 = -\frac{\pi^2}{4}\left(k_2\frac{b}{a}+k_1\frac{a}{b}\right),\\
I_4 = \frac{4}{3}\frac{\pi^2}{ab},\\
I_5 = -\frac{\pi^2}{4}\left(k_2\frac{b}{a}+k_1\frac{a}{b}\right),\\
I_6 = \frac{4ab}{A\pi^2}[k_2p_0(1-0{,}5\xi)+k_1r_0(1-0{,}5\eta)],\\
I_7 = \frac{\pi^4}{4}\left(\frac{b}{a}+\frac{a}{b}\right)^2\frac{1}{ab},\\
I_8 = \frac{8}{3}\frac{\pi^2}{ab},
\end{gathered} \tag{2.18}$$

$$I_9 = -\frac{\pi^2}{4A}\left[\frac{b}{a} r_0 (1 - 0{,}5\eta) + \frac{a}{b} p_0 (1 - 0.5)\xi\right],$$

$$I_{10} = \int_0^b \int_0^a q(x, y) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}\, dx\, dy. \qquad \text{(2.18 continued)}$$

Introducing the obtained values of $I_i$ in (2.9), we will obtain:

$$A = \frac{3Ef(k_1 a^2 + k_2 b^2) - 16Ef^3}{3\pi^2\left(\frac{b}{a} + \frac{a}{b}\right)^2}. \qquad (2.19)$$

The general solution (2.10) after substitution there of values from (2.18) and performance of calculations will assume the following form

$$\begin{aligned} q_a^* - \frac{16}{\pi^4}[\varkappa_2 p_0^*(1 - 0{,}5\xi) + \varkappa_1 r_0^*(1 - 0{,}5\eta)] + \\ + [r_0^*(1 - 0{,}5\eta) + p_0^*(1 - 0{,}5\xi)]\zeta = \frac{\pi^2}{12(1-\nu^2)}\left(\gamma + \frac{1}{\gamma}\right)^2\zeta + \\ + \frac{(\varkappa_1 + \varkappa_2)^2}{\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}\zeta - \frac{16(\varkappa_1 + \varkappa_2)}{\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}\zeta^2 + \frac{512}{9\pi^6\left(\gamma + \frac{1}{\gamma}\right)^2}\zeta^3. \end{aligned} \qquad (2.20)$$

[Л = a = any]

Here we introduce dimensionless parameters: $q_a^* = \frac{4ab}{\pi^2 Eh^1} I_{10}$ is the parameter of any transverse load.

In the particular case of an evenly-distributed load:

$$q_a^* = \frac{16}{\pi^4}\gamma^2 q^*, \quad q^* = \frac{qa^4}{Eh^4}; \quad \text{while} \quad p_0^* = \frac{p_0 a^2}{Eh^2},$$

$r_0^* = \frac{r_0 b^2}{Eh^2}$ are parameters of compressing forces, $\varkappa_1 = \frac{k_1 a^2}{h}$; $\varkappa_2 = \frac{k_2 b^2}{h}$ are parameters of curvature; $\zeta = \frac{f}{h}$ is the relative sag; $\gamma = \frac{b}{a}$ is the relation of sides of the contour, close to a unity.

In these designations expression (2.19) will be written thus:

$$A^* = \frac{3E(\varkappa_1 + \varkappa_2)\zeta - 16E\zeta^3}{3\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}, \qquad (2.21)$$

where

$$A^* = \frac{A}{h^3}.$$

Formula (2.20) yields the sought relationship between the transverse load q(x, y), compressing stresses, and the sag in the center of the shell, secured on a rectangular contour.

In the case of a rectangular plane plate the general solution (2.11) after introduction of values $I_1$ from (2.18) into it will assume the following form:

$$q_n^* + [r_0^*(1-0{,}5\eta) + p_0^*(1-0{,}5\xi)]\zeta = \\ = \frac{\pi^4}{12(1-\nu^2)}\left(\gamma+\frac{1}{\gamma}\right)^2\zeta + \frac{512}{9\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^3. \tag{2.22}$$

The value of parameter A* for the plate will be

$$A^* = -\frac{16E\zeta^3}{3\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}.$$

Now let us investigate the stability of a panel of a cylindrical shell.

Let us assume that on the edge of the shell, which has the shape of a cylindrical panel, act only compressing stresses $r(y) = r_0(1 - \eta \frac{y}{b})$, directed along the generator (Fig. 32). In this case

$$q_n^* = p_0^* = \varkappa_1 = 0.$$

Formulas (2.21) and (2.20) will turn into the following:

$$A^* = \frac{3E\varkappa_2\zeta - 16E\zeta^3}{3\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2},$$

$$r_0^*(1-0{,}5\eta)\zeta = \frac{\pi^4}{12(1-\nu^2)}\left(\gamma+\frac{1}{\gamma}\right)^2\zeta + \frac{\varkappa_2^2}{\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta - \\ - \frac{16\varkappa_2}{\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^2 + \frac{512}{9\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^3. \tag{2.24}$$

Assuming that $\zeta \neq 0$, i.e., examining the panel after the loss of stability, we will obtain:

$$r_0^* = \frac{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}{12(1-\nu^2)}\frac{1}{1-0{,}5\eta_1} + \frac{512}{9\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\frac{1}{1-0{,}5\eta_1}\zeta^2 + $$
$$+ \frac{\kappa_2^2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\frac{1}{1-0{,}5\eta_1} - \frac{16\kappa_2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\frac{1}{1-0{,}5\eta_1}\zeta. \tag{2.25}$$

In the case of a square panel ($\gamma = 1$), compressed by an evenly-distributed load ($\eta = 0$), we will obtain from formula (2.25):

$$r_0^* = \frac{\pi^2}{3(1-\nu^2)} + \frac{128}{9\pi^2}\zeta^2 + \frac{\kappa_2^2}{4\pi^2} - \frac{4\kappa_2}{\pi^2}\zeta. \tag{2.26}$$

This formula of M. A. Koltunov differs little from the result obtained by A. S. Vol'mir [4]:

$$r_0^* = \frac{\pi^2}{3(1-\nu^2)} + \frac{\pi^2}{8}\zeta^2 + \frac{\kappa_2^2}{4\pi^2} - \frac{10\kappa_2}{3\pi^2}\zeta, \tag{2.26'}$$

which is obtained on the assumption of the free slipping of contour points of the shell along the contour.

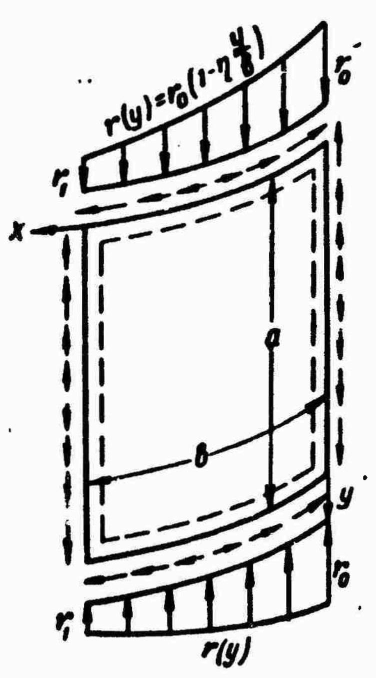



Fig. 32.

In the interval of sags $0 \leq \zeta \leq 4$ formula (2.26) gives a somewhat smaller load for the achievement of the same sag, than formula (2.26').

Graphs of curves (2.26) and (2.26') are drawn in Fig. 33. Assuming that in (2.25) $\zeta = 0$, we will obtain the value of the critical stress:

$$r_в^* = \frac{\pi^2}{12(1-\nu^2)}\left(\gamma+\frac{1}{\gamma}\right)^2\frac{1}{1-0{,}5\eta_1} + \frac{\kappa_2^2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\frac{1}{1-0{,}5\eta_1}. \tag{2.27}$$

[в = u = upper]

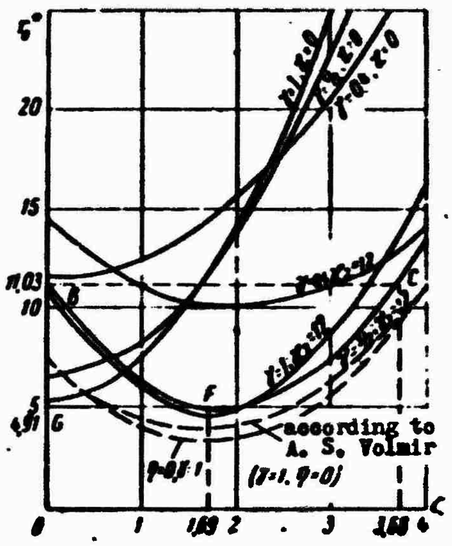



Fig. 33.

which, when $\eta = 0$, coincides accurately with the known S. P. Timoshenko's formula.

Value $r_u^*$ is termed the upper critical stress. For the determination of the lower critical stress from equation

$$\frac{dr_0^*}{d\zeta} = \frac{1024}{9\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\,\frac{1}{1-0.5\eta}\,\zeta - \frac{16\kappa_2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\,\frac{1}{1-0.5\eta} = 0$$

we find the parameter of the relative sag

$$\zeta_0 = \frac{9}{64}\kappa_2, \qquad (2.28)$$

which corresponds to the minimum value of the compressing load. Introducing (2.28) into (2.25), we will obtain the parameter of the lower critical stress:

$$r_{\text{н}}^* = \frac{\pi^2}{12(1-\nu^2)}\left(\gamma+\frac{1}{\gamma}\right)^2\frac{1}{1-0.5\eta} - \frac{\kappa_2^2}{8\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\,\frac{1}{1-0.5\eta}. \qquad (2.29)$$

[H = 1 = lower]

For the square panel this formula, when $\eta = 0$, gives

$$(r_{\text{н}}^*)_{\text{кв}} = \frac{\pi^2}{3(1-\nu^2)} - \frac{\kappa_2^2}{32\pi^2}, \qquad (2.30)$$

[KB = sq = square]

which is somewhat less than A. S. Vol'mir's result:

$$(r_{\text{н}}^*)_{\text{кв}} = \frac{\pi^2}{3(1-\nu^2)} - \left(\frac{200}{9\pi^4} - \frac{0.25}{\pi^2}\right)\kappa_2^2.$$

Permissible loads must be selected in such a manner that the safety factor would be assured with respect to the lower critical stress. With such a selection of critical stresses we remove the

possibility of appearance of flapping of the panel in the process of operation of the thin-walled structure.

In formula (2.25) the plus sign of the parameter of sag corresponds to the sag in the direction of the center of curvature, the minus sign — to the sag directed from the center of curvature. From equation (2.26) it is clear that upon the loss of stability the sag of the shell should be directed toward the center of the curvature, since the increase of the dip of the sag in the opposite direction is connected with a rapid increase of the compressing load. In Fig. 33 are the graphs of $r^*(\zeta)$. Formulas (2.26) for $\gamma = 0.4$; 1.0 and 1.5 as applied to cylindrical panels with the parameter of curvature $\varkappa_2 = 12$ and for plane plates with that ratio of sides, with $\eta = 2/3$.

Upon examination of these graphs we can see that in plates, after the loss of stability, the growth of the sag is connected with the increase of the load.

For shells we have a somewhat more complicated picture. The least increase of the upper critical stress (when $\zeta = 0$) is connected with a very rapid increase of the sag, which is accompanied, after the loss of stability, by a decrease of applied external load. An abrupt sag obtains the value $\zeta_1 \neq 0$, to which corresponds a load, equal in size to value $r_u^*$. The curvature of the shell meanwhile will change its sign. Thus, we have here the phenomenon of flapping. Thus, for instance, a shell with parameters of curvature $\varkappa_2 = 12$; $\gamma = 3/2$ and $\eta = 2/3$ after achieving the upper critical stress $r_u^* = 6.364 + 4.662 = 11.03$ loses stability, and the sag abruptly attains the value $\zeta_1 = 3.68$, corresponding to point C of the curve. Further growth of sag is connected with an increase of the load. In the interval $0 \leq \zeta \leq 3.68$ there exists a point $\zeta_0 = 9/64$, $\varkappa_2 = 1.692$, to

which corresponds the minimum "load" $r_1^* = 4.91$ (point F of the curve). Deflection of sag in any direction from value $\zeta_0$ is connected with an increase of the compressing load. The region, limited by horizontals BC and GF, is termed the region of instability of the shell. Its upper and lower boundaries correspond to the upper and lower critical stresses.

Let us note that in the case of the plate ($\varkappa_2 = 0$) the critical stress, obtained from (2.25) and (when $\zeta = 0$):

$$(r^*_{пл})_{кр} = \frac{\pi^2}{12(1-\nu^2)}\left(\gamma + \frac{1}{\gamma}\right)^2 \frac{1}{1-0.5\eta},$$

[пл = pl = plate; кр = cr = critical]

coincides exactly with the known formula of S. P. Timoshenko.

Let us consider a shell of any Gaussian curvature under the action of an arbitrary transverse load.

Let us assume that no external forces are applied to the edges of the shell. Then, assuming that $p_0^* = r_0^* = 0$, from formula (2.20) we will obtain:

$$q_a^* = \frac{\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}{12(1-\nu^2)}\zeta + \frac{(\varkappa_1+\varkappa_2)^2}{\pi^6\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta - \frac{16(\varkappa_1+\varkappa_2)}{\pi^6\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^2 + \frac{512}{9\pi^6\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^3. \tag{2.31}$$

This relationship yields the relationship between any transverse load $q(x, y)$ and sag $f = \zeta h$ in the center of shell, supported by a rigid contour with a rectangular plan (Fig. 34).

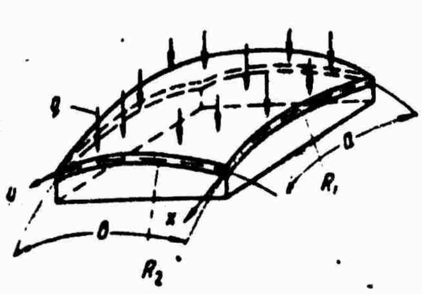


Fig. 34.

In the particular case of an evenly-distributed load q = const

$$I_{10} = \int_0^b\int_0^a q \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}\, dx\, dy = \frac{4ab}{\pi^2} q. \tag{2.32}$$

and formula (2.31) upon multiplication of both parts by $\frac{\pi^4}{16\gamma^2}$, will assume the form

$$q^* = \frac{\pi^4\left(1+\frac{1}{\gamma^2}\right)^2}{192(1-\nu^2)}\zeta + \frac{\pi^2(\varkappa_1+\varkappa_2)^2}{16(1+\gamma^2)^2}\zeta - \frac{\pi^2(\varkappa_1+\varkappa_2)}{(1+\gamma^2)^2}\zeta^2 + \frac{32\pi^2}{9(1+\gamma^2)^2}\zeta^3. \tag{2.33}$$

For linear load $q(x, y) = \frac{q_0}{a}x$; $I_{10} = \frac{2ab}{\pi^2}q_0$, and solution (2.31) in this case will be

$$q_0^* = \frac{\pi^4\left(1+\frac{1}{\gamma^2}\right)^2}{96(1-\nu^2)}\zeta + \frac{\pi^2(\varkappa_1+\varkappa_2)^2}{8(1+\gamma^2)^2}\zeta - \frac{2\pi^2(\varkappa_1+\varkappa_2)}{(1+\gamma^2)^2}\zeta^2 + \frac{64\pi^2}{9(1+\gamma^2)^2}\zeta^3. \tag{2.34}$$

Formulas (2.31), (2.33), (2.34) are true for shells of any Gaussian curvature $\Gamma = k_1k_2$.

<u>The shells of zero Gaussian curvature</u> (cylindrical panel). Let us assume that on such a shell acts a transverse load q evenly distributed on its convex surface. Assuming in formula (2.33) that $\varkappa_1 = 0$, we will obtain the solution for this case:

$$q^* = \frac{\pi^4\left(1+\frac{1}{\gamma^2}\right)^2}{192(1-\nu^2)}\zeta + \frac{\pi^2\varkappa_2^2}{16(1+\gamma^2)^2}\zeta - \frac{\pi^2\varkappa_2}{(1+\gamma^2)^2}\zeta^2 + \frac{32\pi^2}{9(1+\gamma^2)^2}\zeta^3. \tag{2.35}$$

For a square panel ($\gamma = 1$) this formula will assume the form

$$q_{\varkappa 2}^* = \frac{\pi^4}{48(1-\nu^2)}\zeta + \frac{\pi^2\varkappa_2^2}{64}\zeta - \frac{\pi^2\varkappa_2}{4}\zeta^2 + \frac{8\pi^2}{9}\zeta^3. \tag{2.36}$$

Let us find the boundaries of the region of stability in this

case. From equation

$$\frac{dq^*}{d\zeta} = \frac{\pi^6\left(1+\frac{1}{\gamma^2}\right)^2}{192(1-\nu^2)} + \frac{\pi^6\varkappa_2}{16(1+\gamma^2)^2} - \frac{2\pi^6\varkappa_2^2}{(1+\gamma^2)^2}\zeta +$$
$$+\frac{32\pi^6}{3(1+\gamma^2)^2}\zeta^2 = 0,$$

obtained by differentiation of (2.35), we will establish the value of relative sags $\zeta_0$ and $\zeta_1$, corresponding to the extremum of the load parameter $q^*$. Solving the equation, we will obtain:

$$\zeta_{0,1} = \frac{3}{32}\varkappa_2 \mp \frac{1}{32}\sqrt{3\varkappa_2^2 - \frac{\pi^4\left(\gamma+\frac{1}{\gamma}\right)^4}{2(1-\nu^2)}}. \tag{2.37}$$

If the subradical expression in (2.37) turns into zero, then values $\zeta_0$ and $\zeta_1$ coincide. Consequently, shells with parameters of curvature

$$\varkappa_2 = \frac{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}{\sqrt{6(1-\nu^2)}} \tag{2.38}$$

have only one parameter of sag

$$\zeta' = \frac{3}{32}\varkappa_2, \tag{2.39}$$

which corresponds to the extremum of the load. It is not difficult to see that at point $\zeta_1$ curve $q^*(\zeta)$ has a point of inflection.

Introducing by turns values $\zeta_0$ and $\zeta_1$ from (2.37) in (2.35), we will obtain values of extreme parameters of load $q_0^*$ and $q_1^*$, which coincide when $\zeta' = \frac{3}{32}\varkappa_2$.

Fig. 35 shows graphs of dependencies $q^*(\zeta)$ in the case of the square panel (formula 2.36) for certain values of parameters of curvature $\left(\varkappa_2 = 0; 6; \frac{4\pi^2}{\sqrt{6(1-\nu^2)}}; 12\right)$. In Table 5 we adduce values of

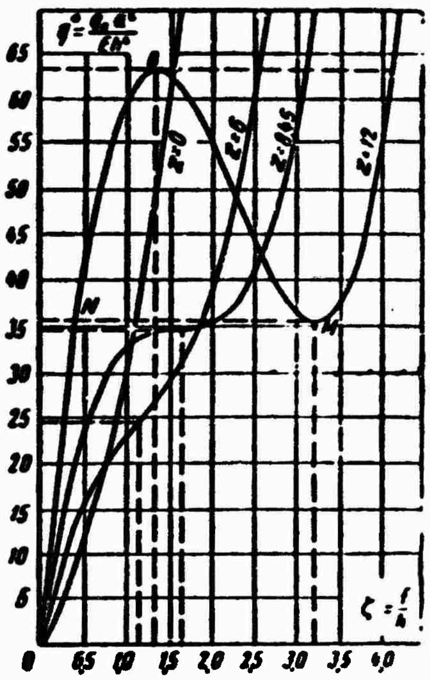



Fig. 35.

parameters of load q* for values of relative sags $0 \le \zeta \le 4$.

<u>The shell of positive Gaussian curvature.</u> Let us examine the behavior of a spherical shell ($k_1 = k_2$), which is under the effect of an evenly distributed load q and is supported by a contour rectangular in the plan. For the spherical shell ($k_1 = k_2 = k$) we have $\varkappa_2 = \gamma^2 \varkappa_1$, which ensues from the fact $\varkappa_1 = \frac{ka^2}{h}$; $\varkappa_2 = \frac{kb^2}{h}$. Introducing this in (2.33), we will obtain solution:

$$q^* = \frac{\pi^4\left(1+\frac{1}{\gamma^2}\right)^2}{192(1-\nu^2)}\zeta + \frac{\pi^2\varkappa_1^2}{16}\zeta - \frac{\pi^2\varkappa_1}{1+\gamma^2}\zeta^2 + \frac{32\pi^2}{9(1+\gamma^2)^2}\zeta^3. \qquad (2.40)$$

Values of parameters $\zeta$, which correspond to extreme values of the load parameter q*, will be determined from equation $\frac{dq^*}{d\zeta} = 0$. We obtain:

$$\zeta_{0;1} = \frac{3\varkappa_1(1+\gamma^2)}{32} \pm \frac{1}{32}\sqrt{3\varkappa_1^2(1+\gamma^2)^2 - \frac{\pi^4\left(\gamma+\frac{1}{\gamma}\right)^4}{2(1-\nu^2)}}, \qquad (2.41)$$

where for shells with parameter of curvature

$$\varkappa > \frac{\pi^2\left(1+\frac{1}{\gamma^2}\right)}{\sqrt{6(1-\nu^2)}} \qquad (2.42)$$

point $\zeta_0$ and $\zeta_1$ coincide. Such shells have only one extreme value for the load parameter q*, corresponding to the sag parameter:

$$\zeta' = \frac{3\varkappa_1(1+\gamma^2)}{32}.$$

In the case of a square panel of a spherical shell ($\gamma = 1$)

formulas (2.40)-(2.41) will be transformed into the following:

$$q^* = \frac{\pi^6}{48(1-\nu^2)}\zeta + \frac{\pi^2 k_1^2}{16}\zeta - \frac{\pi^2 k_1}{2}\zeta^2 + \frac{8\pi^2}{9}\zeta^3; \tag{2.43}$$

$$\zeta_0 = \frac{3}{16}k_1 - \frac{1}{16}\sqrt{3k_1^2 - \frac{2\pi^4}{1-\nu^2}}; \tag{2.44}$$

$$\zeta_1 = \frac{3}{16}k_1 + \frac{1}{16}\sqrt{3k_1^2 - \frac{2\pi^4}{1-\nu^2}}. \tag{2.45}$$

Introducing values $\zeta$ from (2.44) and (2.45) in (2.43), we will obtain extreme load parameters:

$$q^*_{\text{в}} = \frac{\pi^6 k_1}{256(1-\nu^2)} + \frac{\pi^2}{256}\left[\frac{k_1^2}{2} - \frac{\pi^4}{3(1-\nu^2)}\right]\sqrt{3k_1^2 - \frac{2\pi^4}{1-\nu^2}} - \frac{\pi^2}{1608}\sqrt{\left(3k_1^2 - \frac{2\pi^4}{1-\nu^2}\right)^3}; \tag{2.46}$$

$$q^*_{\text{н}} = \frac{\pi^6 k_1}{256(1-\nu^2)} - \frac{\pi^2}{256}\left[\frac{k_1^2}{2} - \frac{\pi^4}{3(1-\nu^2)}\right]\sqrt{3k_1^2 - \frac{2\pi^4}{1-\nu^2}} - \frac{\pi^2}{1608}\sqrt{\left(3k_1^2 - \frac{2\pi^4}{1-\nu^2}\right)^3}. \tag{2.47}$$

For the shell with curvature parameter:

$$k_1 = \frac{2\pi^2}{\sqrt{6(1-\nu^2)}} \cong 8{,}45 \quad \text{when } \nu = 0{,}3, \tag{2.48}$$

which corresponds to the radius of curvature

$$R = \frac{\sqrt{6(1-\nu^2)}}{2\pi^2}\frac{a^2}{h} \cong 0{,}12\frac{a^2}{h}, \tag{2.49}$$

sag parameters $\zeta_0$ and $\zeta_1$ coincide

$$\zeta' = \frac{3}{16}k_1 = 1{,}584, \tag{2.50}$$

and corresponding extreme value of the load parameter will be

$$q^*_{\text{в}} = q^*_{\text{н}} = q^* = \frac{\pi^6 k_1}{256(1-\nu^2)} \cong 34{,}876 \quad (\text{for } \nu = 0{,}3), \tag{2.51}$$

or, since $q^* = \frac{qa^4}{Eh^4}$, then $q_{extr} \cong 34.9 \frac{Eh^4}{a^4}$.

Shells of such type are threshold cases between shells and plates. Their behavior under load, as it can be seen from Fig. 35, is somewhat different from the behavior of plates and cannot be characterized as behavior of a shell, which has a curvature parameter larger than that given by formula (2.48). Dependency $q^*(\zeta)$ for such shells will have the following form

$$q^* = \frac{\pi^6}{16(1-\nu^2)}\zeta - 4{,}224\pi^2\zeta^2 + \frac{8\pi^2}{9}\zeta^3. \tag{2.52}$$

In Fig. 35 graphs are plotted for dependencies $q^*(\zeta)$ for values of parameters of curvature $\varkappa_1 = 0;\ 6;\ 8.45,\ 12$ (when $\nu = 0.3$).

From the examination of these graphs we conclude that:

a) Sags of plate ($\varkappa_1 = 0$) increase with the growth of the load according to the nonlinear law, which is expressed by relationship

$$q^* = \frac{\pi^6}{192(1-\nu^2)}\left(1 + \frac{1}{\gamma^2}\right)^2\zeta + \frac{32\pi^2}{9(1+\gamma^2)^2}\zeta^3, \tag{2.53}$$

where in the case of a square panel one should consider $\gamma = 1$.

b) Sags of shells with extremely small curvature ($\varkappa_1 < 8.45$) also increase with the growth of the load. However curve $q^*(\zeta)$ has a point of inflection, which corresponds to the change in the process of deformation of the sign of the shell's curvature. This change proceeds smoothly, without sharp increases of sags, when the load is increased slowly. In this case parameters $\zeta'$ and $q^*$, corresponding to the point of inflection, will be

$$\zeta' = 1{,}125; \quad q^* = 24{,}76.$$

c) Sags of shell with a curvature parameter (2.48) increase slowly with the increase of load to a certain limit (2.51). Upon

attaining value $\zeta' = 1.584$, the sags begin to increase smoothly but rapidly in a certain interval even with an insignificant increase of the load. In point $\zeta' = 1.584$ there exists a state of neutral equilibrium. Further growth of sags of such a shell, which has already changed the sign of the curvature, is related to a rapid increase of the load.

d) Sags of a shell with curvature parameter $\varkappa_1 > 8.45$ (here on the curve of Fig. 35 $\varkappa_1 = 12$) increase slowly with the increase of the load to a value $\zeta_0 = 1.33$, which corresponds to the upper critical stress $q_b^* = 63.3$ (see formula (2.46)). After attaining value $\zeta_0$ the growth of the sag can continue even with a decrease of load parameter $q^*$. This indicates the fact that in point B of curve $q^*(\zeta)$, i.e., when $\zeta_0 = 1.33$, the form of equilibrium of the shell becomes unstable, and the shell bulges.

Upon the least increase of load $q_b^* = 63.3$ sag $\zeta$ jumps from value $\zeta_0$ to value $\zeta_1$. On the graph this new form of equilibrium is marked by point C. This form is stable, and a further increase of load is accompanied by a gradual increase of sag of the shell, the sign of curvature of which has already changed. This is the phenomenon of snapping of the shell. If beginning with point C we will decrease the load, then the sag of the shell will gradually decrease to a value of $\zeta = 3.17$ ($q_1^* = 35.75$). In this case "load" $q_1^* = 35.75$ will no longer be sufficient to preserve the center of the downward curvature and the shell snaps upwards to the position, marked on the graph by point N. Thus, for spherical shells with curvature parameter $\varkappa_1 >$ $> 8.45$, supported on a contour which is square in the plan, there exists a region, limited in Fig. 35 (for $\varkappa_1 = 12$) by dotted lines BC and MN, inside which the shell has two forms of equilibrium. This

region gradually narrows with the decrease of curvature parameter $\varkappa_1$. When $\varkappa_1 = 8.45$ line BC and MN merge, and when $q^* = 34.9$ we have a neutral equilibrium. When $\varkappa_1 < 8.45$ only one form of equilibrium will be possible. In Table 5 we adduce values of parameters of load $q^*$ for $0 \leq \zeta \leq 4$ in the case of loading of a square panel of a spherical shell by an evenly distributed load. It is easy to note that these graphs for the spherical panel coincide with graphs for the cylindrical panel, the curvature parameter of which is twice as large as curvature parameters of the spherical square panel.

Let us investigate compressed bent sloping shells of negative Gaussian curvature.

Let us examine a shell with Gaussian curvature $\Gamma = k_1 k_2 < 0$, to the edges of which compressing stresses $p(x) = p_0$ are applied, and an evenly distributed load $q(x, y) = q$ acts normally to the middle surface. To be specific we will assume $k_1 > 0$, $k_2 < 0$ (Fig. 36) while we assume that $|k_1| \leq |k_2|$ which in dimensionless parameter can be written thus: $\gamma^2 |\varkappa_1| \leq |\varkappa_2|$.

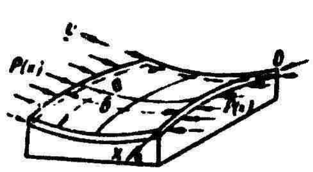


Fig. 36.

Let us assume that

$$\varkappa_2 = \delta\gamma^2\varkappa_1, \text{ where } \delta = \frac{k_2}{k_1} < 0. \tag{2.54}$$

We will introduce (2.54) into (2.20). Taking into consideration that $r_0^* = \xi = \eta = 0$ we will obtain:

$$q^* - \delta\varkappa_1 p_0^* + \frac{\pi^4}{16\gamma^2} p_0^* \zeta = \frac{\pi^4}{192(1-\nu^2)}\left(1 + \frac{1}{\gamma^2}\right)^2 \zeta + \frac{\pi^2\varkappa_1^2(1+\delta\gamma^2)^2}{16(1+\gamma^2)^2}\zeta - \frac{\pi^2\varkappa_1(1+\delta\gamma^2)}{(1+\gamma^2)^2}\zeta^2 + \frac{32\pi^2}{9(1+\gamma^2)^2}\zeta^3. \tag{2.55}$$

If the transverse load is absent ($q^* = 0$), then we will have a case of compression of shell "by stress" $p_0^*$, the dependence of which on

Table 5. Values of Load $q_b^*$ Parameters.

| Curvature κ: cylindrical panel | Curvature κ: spherical panel | $c$ = 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2,210 | 4,472 | 6,840 | 9,365 | 12,102 | 15,101 | 18,416 |
| 12 | 6 | 4,134 | 7,729 | 10,837 | 13,511 | 15,803 | 17,766 | 19,452 |
| $\frac{4\kappa^2}{\sqrt{6(1-\nu^2)}}$ | $\frac{2\kappa^2}{\sqrt{6(1-\nu^2)}}$ | 6,195 | 11,608 | 16,293 | 20,303 | 23,689 | 26,507 | 28,801 |
| 24 | 12 | 10,500 | 19,829 | 28,158 | 35,421 | 41,710 | 47,076 | 51,578 |

Continuation Table 5

| Curvature κ: cylindrical panel | Curvature κ: spherical panel | 0.8 | 0.9 | 1.0 | 1.2 | 1.3 | 1.5 | 1.6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 22,100 | 26,204 | 30,783 | 41,571 | 54,887 | 62,723 | 71,150 |
| 12 | 6 | 20,915 | 22,207 | 23,381 | 25,582 | 27,943 | 29,314 | 30,882 |
| $\frac{4\kappa^2}{\sqrt{6(1-\nu^2)}}$ | $\frac{2\kappa^2}{\sqrt{6(1-\nu^2)}}$ | 30,633 | 32,052 | 33,111 | 34,359 | 34,800 | 34,849 | 34,852 |
| 24 | 12 | 55,261 | 58,182 | 60,392 | 62,889 | 63,177 | 62,723 | 61,675 |

Continuation Table 5

| Curvature κ: cylindrical panel | Curvature κ: spherical panel | 1.8 | 2.0 | 2.2 | 2.4 | 2.5 | 2.6 | 2.8 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 90,782 | 114,20 | 141,84 | 174,10 | 192,10 | 211,42 | 254,21 |
| 12 | 6 | 34,821 | 40,181 | 47,384 | 56,850 | 62,564 | 69,001 | 84,257 |
| $\frac{4\kappa^2}{\sqrt{6(1-\nu^2)}}$ | $\frac{2\kappa^2}{\sqrt{6(1-\nu^2)}}$ | 34,938 | 35,478 | 36,895 | 39,608 | 41,583 | 44,039 | 50,609 |
| 24 | 12 | 58,804 | 54,986 | 50,641 | 46,191 | 44,058 | 42,057 | 38,660 |

Continuation Table 5

| Curvature κ: cylindrical panel | Curvature κ: spherical panel | 3.0 | 3.2 | 3.4 | 3.5 | 3.6 | 3.8 | 4.0 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 302,90 | 357,90 | 419,65 | 453,18 | 488,55 | 565,03 | 649,51 |
| 12 | 6 | 103,04 | 125,77 | 152,87 | 168,19 | 184,76 | 221,86 | 254,60 |
| $\frac{4\kappa^2}{\sqrt{6(1-\nu^2)}}$ | $\frac{2\kappa^2}{\sqrt{6(1-\nu^2)}}$ | 59,739 | 71,850 | 87,364 | 96,528 | 106,70 | 130,28 | 158,53 |
| 24 | 12 | 36,421 | 35,761 | 37,101 | 38,643 | 40,862 | 47,466 | 57,334 |

sags $\zeta$ will have the form (we assume that $\zeta \neq 0$)

$$p_0^*\zeta - \frac{16}{\pi^4}\delta\gamma^2\varkappa_1 p_0^* = \frac{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}{12(1-\nu^2)}\zeta + \frac{\varkappa_1^2(1+\delta\gamma^2)^2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta - \frac{16\varkappa_1(1+\delta\gamma^2)}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^2 + \frac{512}{9\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}\zeta^3. \qquad (2.56)$$

In the case square panel of a pseudosphere, for which one should assume that $\gamma = 1$; $\delta = -1$, formula (2.56) will be converted into the following:

$$p_0^*\zeta + \frac{16}{\pi^4}\varkappa_1 p_0^* = \frac{\pi^2}{3(1-\nu^2)}\zeta + \frac{128}{9\pi^2}\zeta^3 \quad \text{or}$$

$$p_0^*\zeta - \frac{16}{\pi^4}\varkappa_1 p_0^* = \frac{\pi^2}{3(1-\nu^2)}\zeta + \frac{128}{9\pi^2}\zeta^3. \qquad (2.57)$$

If compressing stresses are absent ($p_0^* = 0$), then from (2.55) we will obtain:

$$q^* = \frac{\pi^6\left(1+\frac{1}{\gamma^2}\right)^2}{192(1-\nu^2)}\zeta + \frac{\pi^2\varkappa_1^2(1+\delta\gamma^2)^2}{16(1+\gamma^2)^2}\zeta - \frac{\pi^2\varkappa_1(1+\delta\gamma^2)}{(1+\gamma^2)^2}\zeta^2 + \frac{32\pi^6}{9(1+\gamma^2)^2}\zeta^3. \qquad (2.58)$$

For the square panel of the pseudosphere ($\gamma = 1$, $\delta = -1$) from (2.58) we have:

$$q^* = \frac{\pi^6}{48(1-\nu^2)}\zeta + \frac{8\pi^6}{9}\zeta^3. \qquad (2.59)$$

Formula (2.59) represents a dependency between load and sag in the center of a square plate (see 2.53). Consequently, the square panel of the pseudosphere loaded only with a transverse load, behaves as a square plate, independently of values of the shell curvatures. Setting up derivative $\frac{dq^*}{d\zeta}$ of function (2.58) and equating it to zero, we will obtain the equation, from which we will find the following

values of relative sags, corresponding to the extreme parameters of the load:

$$\zeta_{0;1}=\frac{3}{32}k_1(1+\delta\gamma^2)\pm\frac{1}{32}\sqrt{3k_1^2(1+\delta\gamma^2)^2-\frac{\pi^4}{2(1-\nu^2)}\left(\gamma+\frac{1}{\gamma}\right)^4}. \tag{2.60}$$

These points coincide for shells with parameters of curvature:

$$k_1(1+\delta\gamma^2)=\frac{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}{\sqrt{6(1-\nu^2)}}. \tag{2.61}$$

From examination of formulas (2.55), (2.56) and (2.57) it follows that shells of negative Gaussian curvature, as well as shells of positive Gaussian curvatures, subjected to compression by forces applied to their edges, do not have a region of stable equilibrium.

This is explained by the presence of a curvature of the shell in the direction of the action of external forces. In the presence of compressing forces a stable initial state of a shell can be assured by application of an appreciable transverse load. Some information on the shell of negative Gaussian curvature can be obtained in the book by V. Flyugge [53].

Let us make certain comments on classification of shells. Such terms encountered in literature as: sloping shell and weakly distorted plate are sometimes treated as having the same meaning. The above examined behavior of shells, subjected to the action of a transverse load, enables us to establish the difference between concepts of weakly distorted plates and sloping shells. Let us find points $\zeta_0$ and $\zeta_1$, corresponding to extreme values of an arbitrary transverse load $q_a^*$. Proceeding from formula (2.31), let us construct equation $\frac{dq_a^*}{d\zeta}=0$, and then find the roots of the quadratic equation obtained.

We have

$$\zeta_{0;1} = \frac{3}{32}(\varkappa_1 + \varkappa_2) \pm \frac{1}{32}\sqrt{3(\varkappa_1 + \varkappa_2) - \frac{\pi^4\left(\gamma + \frac{1}{\gamma}\right)^4}{2(1-\nu^2)}}. \qquad (2.62)$$

Equating the subradical expression to zero, we will obtain the sum of parameters of curvatures

$$\varkappa_1 + \varkappa_2 = \frac{\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}{\sqrt{6(1-\nu^2)}} \qquad (2.63)$$

for those shells, point $\zeta_0$ and $\zeta_1$ for which coincide. It is easy to see that formulas (2.38), (2.42), (2.48) and (2.61) are particular cases of formula (2.63). We saw that for shells with parameters of curvature, given by these formulas, only one form of equilibrium is possible, and dependency $q^*(\zeta)$ is a monotonously increasing function without decrease intervals. If, however, the sum of parameters of curvatures $\varkappa_1 + \varkappa_2$ is larger than the right side of (2.63), then function $q^*(\zeta)$ has the decrease interval of "load" $q^*$ in a certain interval of change $\zeta$ and such shells pop. In connection with this it is possible to classify shells according to their work under load.

1. Shells, in which every curvature is equal to zero, are classified as plates. Their behavior under load is characterized by a monotonously increasing curve.

2. Shells, in which the sum of parameters of main curvatures

$$\varkappa_1 + \varkappa_2 < \frac{\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}{\sqrt{6(1-\nu^2)}},$$

should be related to the class of weakly distorted plates. Their behavior under load is characterized by a monotonously increasing curve, with a point of inflection, where curve $q^*(\zeta)$ changes the sign of its curvature. The sign of equality pertains to shells which are

on the borderline between weakly curved plates and sloping shells.

3. Shells, in which the sum of parameters of main curvatures

$$k_1 + k_2 > \frac{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}{\sqrt{6(1-\nu^2)}},$$

should be called sloping shells, if at the moment of loss of stability on the boundaries of the region of instability no plastic deformation will appear.

And finally, a few general considerations concerning the problem of the accuracy of approximate solutions. The answer to the question, how close to the truth are approximate solutions, can be obtained either by means of comparison with data from experiment, if they are set up with a sufficient precision, or by means of comparison with exact solutions, or, finally, by means of theoretical research of convergence, where we must also show the practical convergence of the solution if it is constructed by methods of approximation in series.

Exact solutions of similar problems, in view of their complexity, are unknown. An exact solution in series for flexible round plates was obtained by Way [54], and also by M. S. Kornishin and Kh. M. Mushtari [47] for the circular cylindrical panel under the action of external normal pressure.

Let us analyze the solution of M. A. Koltunov [49] for the problem about the bend and stability of rectangular panels of sloping flexible shells (Fig. 37) by the Bubnov-Galerkin method in high approximations. The problem of convergence of the method for such problems was studied by Kh. M. Mushtari [47] and I. I. Vorovich [50], from whose works it follows that solution of the problem in series by this method should be convergent. Let us construct a solution of the problem about the bend of a sloping shell, taking one, two, three and four terms of

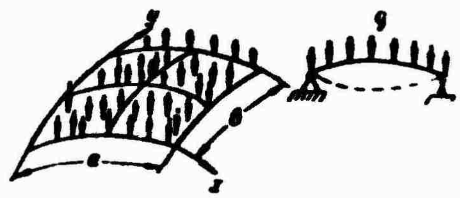


Fig. 37.

series which are approximating the functions of the sag and stresses.

Let us assume that boundary conditions of hinged fastening of edges are the following:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \qquad (x = 0,\ x = a),$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \quad (y = 0,\ y = b). \tag{2.64}$$

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad \tau = -\frac{\partial^2 \varphi}{\partial x\, \partial y} = 0 \quad (x = 0,\ x = a),$$

$$\sigma_y = \frac{\partial^2 \varphi}{\partial x^2} = 0, \quad \tau = -\frac{\partial^2 \varphi}{\partial x\, \partial y} = 0 \quad (y = 0,\ y = b).$$

As approximating functions let us select the following:

$$w = f_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b},$$

$$\varphi = A_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b},$$

in solving the problem in the "first" approximation;

$$w = f_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + f_3 \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b},$$

$$\varphi = A_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + A_3 \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b}$$

for the "second" approximation;

$$w = f_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + f_3 \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b} + f_5 \sin\frac{5\pi x}{a} \sin\frac{5\pi y}{b},$$

$$\varphi = A_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + A_3 \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b} + A_5 \sin\frac{5\pi x}{a} \sin\frac{5\pi y}{b},$$

in solving the problem in the "third" approximation;

$$w = f_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + f_3 \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b} + f_5 \sin\frac{5\pi x}{a} \sin\frac{5\pi y}{b} + $$
$$+ f_7 \sin\frac{7\pi x}{a} \sin\frac{7\pi y}{b},$$

$$\varphi = A_1 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + A_3 \sin\frac{3\pi x}{a} \sin\frac{3\pi y}{b} + A_5 \sin\frac{5\pi x}{a} \sin\frac{5\pi y}{b} + $$
$$+ A_7 \sin\frac{7\pi x}{a} \sin\frac{7\pi y}{b}$$

for solving the problem in the "fourth" approximation.

Such set of functions is selected for the purpose of more fully satisfying the operating conditions of a shell hinge-secured on all the edges under the action of a load evenly distributed on its convex surface. (Here a symmetric snapping is assumed.) Approaching the solution of the problem on a more strict basis, in [55] equations are built for the full system of approximating functions. However, in the case of symmetric snapping of a sloping shell examined, the terms of the series of the type:

$$\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\qquad (m\neq n)$$

do not have an appreciable influence on the solution, which is confirmed by subsequent research by M. A. Koltunov.

It is not difficult to see that these functions satisfy all boundary conditions, where the last one is executed on the "average"

$$\tau_{cp}=\frac{1}{a}\int_0^a\frac{\partial^2\varphi}{\partial x\,\partial y}\,dx=0.$$

Setting up the Bubnov-Galerkin operation:

$$\iint_{(\omega)}\Phi\delta\varphi\,d\omega=0,\qquad \iint_{(\omega)}W\delta w\,d\omega=0,$$

we will obtain a system of algebraic nonlinear equations.

Let us give here only the last system, when the problem was solved in the "fourth" approximation:

$$\begin{gathered}1)\ \frac{\pi^2}{4}\left(\gamma+\frac{1}{\gamma}\right)^2\beta_1-\frac{\alpha_1+\alpha_2}{4}x_1+\frac{4}{3}x_1^2-\frac{8}{5}x_1x_3-\\-\frac{8}{45}x_1x_7-\frac{8}{21}x_1x_5+\frac{324}{35}x_3^2-\frac{200}{21}x_3x_5-\frac{392}{165}x_3x_7+\frac{2500}{99}x_5^2-\\-\frac{9800}{429}x_5x_7+\frac{9604}{195}x_7^2=0.\end{gathered}\qquad(2.65)$$

$$2)\quad \frac{81\pi^4}{4}\left(\gamma+\frac{1}{\gamma}\right)^2\beta_3-\frac{9(\alpha_1+\alpha_2)}{4}x_3-\frac{4}{5}x_1^2+\frac{648}{35}x_1x_3-$$

$$-\frac{200}{21}x_1x_5-\frac{392}{165}x_1x_7+12x_3^2+\frac{648}{11}x_3x_5-\frac{648}{13}x_3x_7+\frac{2500}{91}x_5^2+$$

$$+\frac{392}{3}x_5x_7+\frac{9604}{187}x_7^2=0,$$

$$3)\quad \frac{625\pi^4}{4}\left(\gamma+\frac{1}{\gamma}\right)^2\beta_5-\frac{25(\alpha_1+\alpha_2)}{4}x_5-\frac{4}{21}x_1^2-\frac{200}{21}x_1x_3+$$

$$+\frac{5000}{99}x_1x_5-\frac{9800}{429}x_1x_7+\frac{324}{11}x_3^2+\frac{5000}{91}x_3x_5+\frac{392}{3}x_3x_7+$$

$$+\frac{100}{3}x_5^2+\frac{5000}{51}x_5x_7+\frac{9604}{171}x_7^2=0,$$

$$4)\quad \frac{2401\pi^4}{4}\left(\gamma+\frac{1}{\gamma}\right)^2\beta_7-\frac{49(\alpha_1+\alpha_2)}{4}x_7-\frac{4}{45}x_1^2-\frac{392}{165}x_1x_3-$$

$$-\frac{9800}{429}x_1x_5+\frac{19208}{195}x_1x_7-\frac{324}{13}x_3^2+\frac{392}{3}x_3x_5+\frac{19208}{187}x_3x_7+$$

$$+\frac{2500}{51}x_5^2+\frac{19208}{171}x_5x_7+\frac{196}{3}x_7^2=0,$$

$$5)\quad q_1=\frac{\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}{192(1-\nu^2)}x_1+\frac{\pi^4}{4}\left[\frac{(\alpha_1+\alpha_2)}{4}\beta_1-\frac{8}{3}\beta_1x_1+\right.$$

$$+\frac{8}{5}\beta_1x_3+\frac{8}{21}\beta_1x_5+\frac{8}{45}\beta_1x_7+\frac{8}{5}\beta_3x_1-\frac{648}{35}\beta_3x_3+\frac{200}{21}\beta_3x_5+$$

$$+\frac{392}{165}\beta_3x_7+\frac{8}{21}\beta_5x_1+\frac{200}{21}\beta_5x_3-\frac{5000}{99}\beta_5x_5+\frac{9800}{429}\beta_5x_7+$$

$$\left.+\frac{8}{45}\beta_7x_1+\frac{392}{165}\beta_7x_3+\frac{9800}{429}\beta_7x_5-\frac{19208}{195}\beta_7x_7\right],$$

$$6)\quad q_3=\frac{981\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}{192(1-\nu^2)}x_3+\frac{9\pi^4}{4}\left[\frac{9(\alpha_1+\alpha_2)}{4}\beta_3+\frac{8}{5}\beta_1x_1-\right.$$

$$-\frac{648}{35}\beta_1x_3+\frac{200}{21}\beta_1x_5+\frac{392}{165}\beta_1x_7-\frac{648}{35}\beta_3x_1-24\beta_3x_3-\frac{648}{11}\beta_3x_5+$$

$$+\frac{648}{13}\beta_3x_7+\frac{200}{21}\beta_5x_1-\frac{648}{11}\beta_5x_3-\frac{5000}{91}\beta_5x_5-\frac{392}{3}\beta_5x_7+$$

$$\left.+\frac{392}{165}\beta_7x_1+\frac{648}{13}\beta_7x_3-\frac{392}{3}\beta_7x_5-\frac{19208}{187}\beta_7x_7\right]. \quad \text{(2.65 continued)}$$

$$7)\quad q_7=\frac{49240\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}{192(1-\nu^2)}x_7+\frac{49\pi^4}{4}\left[\frac{49(\alpha_1+\alpha_2)}{4}\beta_7+\frac{8}{45}\beta_1x_1+\right.$$

$$+\frac{392}{165}\beta_1x_3+\frac{9800}{429}\beta_1x_5-\frac{19208}{195}\beta_1x_7+\frac{392}{165}\beta_3x_1-\frac{648}{13}\beta_3x_3-$$

$$-\frac{392}{165}\beta_3x_5-\frac{19208}{187}\beta_3x_7+\frac{9800}{429}\beta_5x_1-\frac{392}{3}\beta_5x_3-\frac{5000}{51}\beta_5x_5-$$

$$\left.-\frac{19208}{171}\beta_5x_7-\frac{19208}{195}\beta_7x_1-\frac{19208}{187}\beta_7x_3-\frac{19208}{171}\beta_7x_5-\frac{392}{3}\beta_7x_7\right].$$

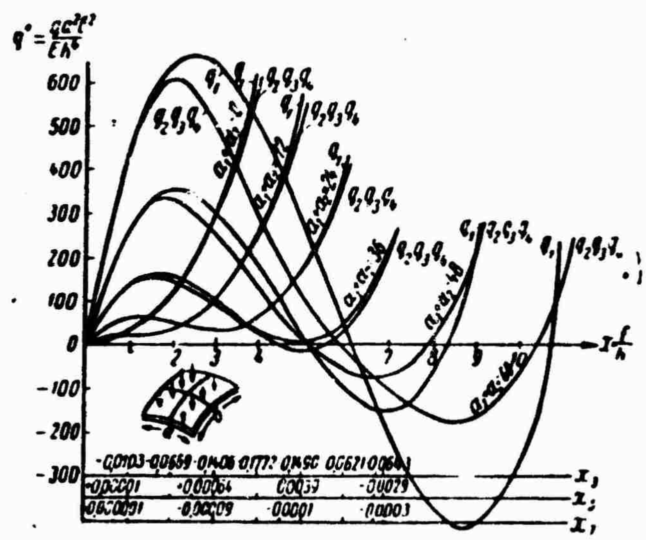
Let us note that coefficients with squares of unknowns can be obtained from the general expression

$$\frac{4\alpha^4}{4\alpha^4-\beta^4}.$$

where i is index $\beta(\beta_{ii})$, n is index with $x^2(x_{nn}^2)$, but with products $x_n x_m$ they do not depend on the order of indices. Here we introduce dimensionless parameters: $\beta_1 = \frac{A_{ji}}{Eh^2}$ is the stress parameter $\gamma = \frac{b}{a}$ is the relationship of sides $x_i = \frac{f_i}{h}$ is the sag parameter, $\alpha_1 = \frac{k_1 a^2}{h}$ $\alpha_2 = \frac{k_2 b^2}{h}$ are parameters of main curvatures; and $q_i = \frac{qa^2b^2}{Eh^4}$ is the parameter of the evenly distributed load.

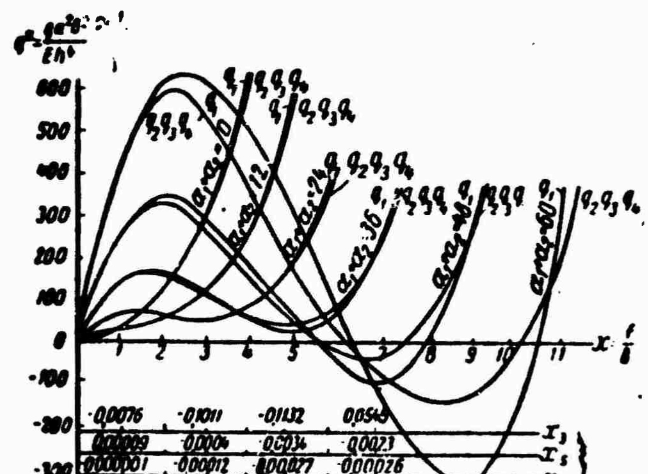
Resolutions of systems of equations of type (2.64) for all four approximations in the form of graphs of load-sag relationship are shown in Figs. 38, 39, and 40. Solutions are obtained on the "Strela" electronic computer.

On all graphs the load-sag curves are built for the first $(q_1)$, second $(q_2)$, third $(q_3)$ and fourth $(q_4)$ approximations. In Table 6 we give the values of loads q for sags $x_i$ and difference $\Delta_{ij}$ between values of parameters of loads in i and j approximations.



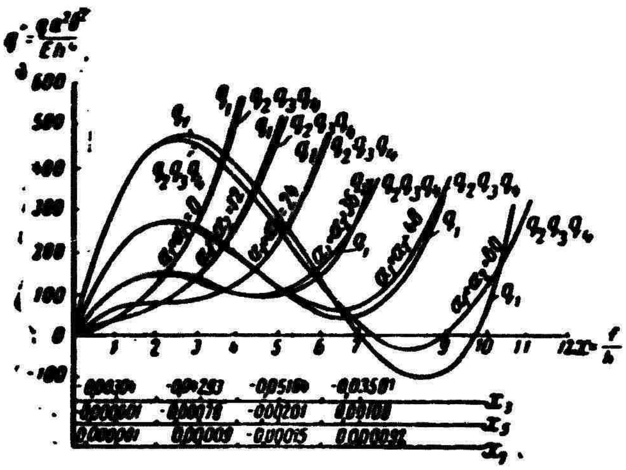

Value $X_3X_5X_7$ for a shell with $\alpha_1 + \alpha_2 = 36$

Fig. 38.



Values $X_3X_5X_7$ for shell with $\alpha_1 + \alpha_2 = 36$

Fig. 39.



Values $X_3 X_5 X_7$ for shell with $\alpha_1 + \alpha_2 = 36$

Fig. 40.

If the difference between the first and second approximations may be significant, especially in regions of adjoining the values of "upper" ($q_u$) and "lower" ($q_l$) critical loads, then the difference between the second and third, third and fourth, second and fourth approximations is insignificant, while, as it is easy to see from the adduced tables, it decreases with the increase of indices i and j. Distinctions of solutions $\Delta_{23}$, $\Delta_{34}$, $\Delta_{14}$ and $\Delta_{24}$ are so small that almost in all graphs plotted for ratios of sides of panels 1, $\sqrt{2}$, and 2 with the sum of parameters of main curvatures from 0 to 60, curves $q_2$, $q_3$, $q_4$ on significant sections of change of sag merge into one.

Table 6. Comparison of "Load" $q_i$, Obtained in 1st-4th Approximations with Difference $\Delta_{ij}$ Between Them for $\alpha_1 + \alpha_2 = 36$.

| | $x_1$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $\Delta_{12}$ | $\Delta_{23}$ | $\Delta_{34}$ | $\Delta_{14}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{b}{a} = 1$ | 1,6 | 163,53 | 160,48 | 160,46 | 160,48 | 1,87% | 0,012% | 0,012% | 1,87% |
| | 5,0 | 14,69 | 1,338 | 1,134 | 1,132 | 9,22% | 15,8% | 0,17% | 93,6% |
| | 6,0 | 28,428 | 48,016 | 48,086 | 48,086 | 68,5% | 0,14% | 0,0% | 69,2% |
| $\frac{b}{a} = \sqrt{2}$ | 1,6 | 153,7 | 151,8 | 151,8 | 151,8 | 1,24% | 0,0% | 0,0% | 1,24% |
| | 5,0 | 12,92 | 24,25 | 24,14 | 24,14 | 87,7% | 0,45% | 0,0% | 86,8% |
| | 5,6 | 26,92 | 42,32 | 42,23 | 42,27 | 57,2% | 0,21% | 0,09% | 57,0% |
| $\frac{b}{a} = 2$ | 2,0 | 142,1 | 141,2 | 141,2 | 141,2 | 0,64% | 0,0% | 0,0% | 0,64% |
| | 4,6 | 90,2 | 93,5 | 93,4 | 93,4 | 3,64% | 0,018% | 0,0% | 3,63% |
| | 5,6 | 112,1 | 119,4 | 119,4 | 119,4 | 6,48% | 0,02% | 0,0% | 6,5% |

Hence we may conclude, that the Bubnov-Galerkin method for the given problem yields a convergent solution, while for practical calculations it is quite sufficient to be limited by solution in the second approximation. In this case it is possible not to resort to the help of electronic computers. It is sufficient, after transformations, to use existing tables of solutions of cubic equations.

Taking into consideration the existing proof of convergence of the Bubnov-Galerkin procedure for these problems, and also extremely small values $\Delta_{ij}$ $(i > 1, j > 1)$, obtained by M. A. Koltunov, we can consider the solutions practically exact. Let us note that M. A. Koltunov obtained solutions of the problem on bending of a round pinched plate in the second approximation by the Bubnov-Galerkin method, which coincide with Way's exact solution [54].

In Fig. 41 we give the graph of dependence of parameters of "upper" and "lower" critical loads on the sum of parameters of main curvatures of shells with the ratio of sides $\gamma = \sqrt{2}$. We also give the corresponding values of parameters of the depth of the sag of all harmonics, included in the fourth approximation, which makes these graphs convenient for calculations of the stressed state of shells in their critical state. For calculation of shells in other states one should use separate graphs of the type (38-40) or special tables. From type 41 graphs it follows that shells will pop, if parameters of their curvatures will be larger than a certain value. In Fig. 41a it is clear that panels of shells with the ratio of sides $\gamma = 1$ will pop, if the sum of parameters of main curvatures

$$\varkappa_1 + \varkappa_2 = \frac{k_1 a^2}{h} + \frac{k_2 b^2}{h} > 18.$$

Let us note that in solving problems in the first approximation we

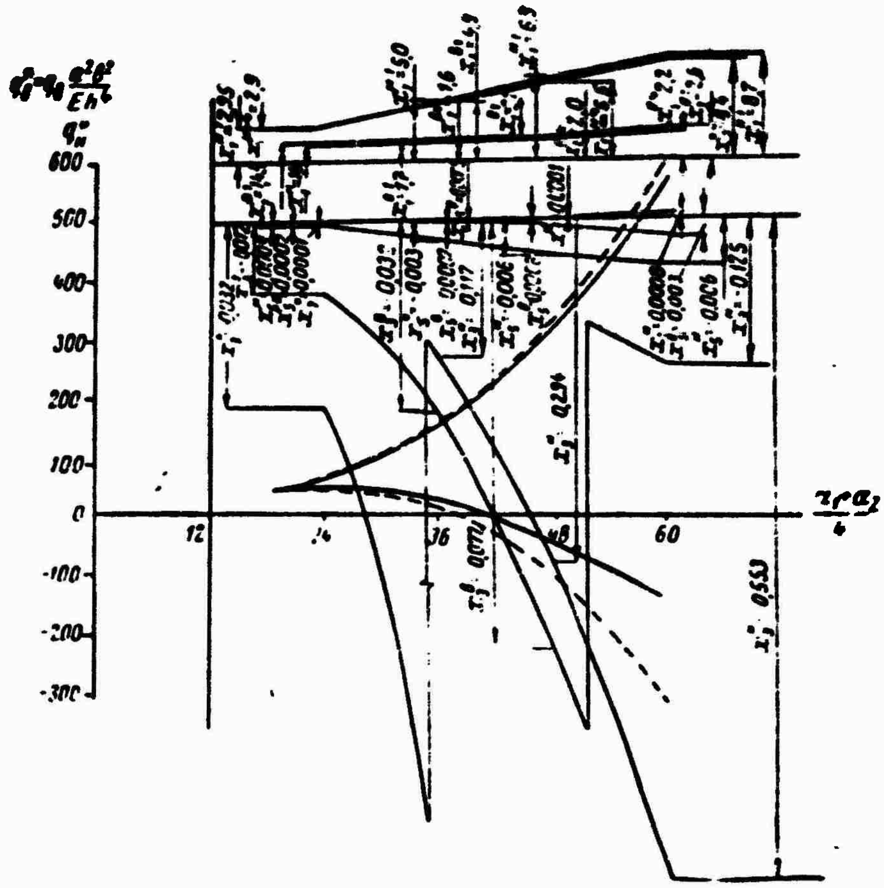


Fig. 41.

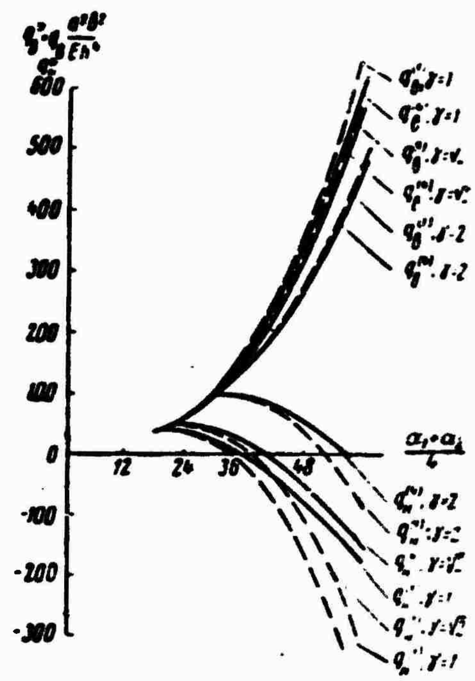


Fig. 41a.

obtained formula (2.63)

$$\alpha_1 + \alpha_2 > \frac{\pi^2\left(\gamma + \frac{1}{\gamma}\right)^2}{\sqrt{6(1-\nu^2)}},$$

which for $\gamma = 1$ yields value $\varkappa_1 + \varkappa_2 = 17$. Panels of shells with $\gamma = \sqrt{2}$, will pop, if $\varkappa_1 + \varkappa_2 \geq 20.4$, and when $\gamma = 2$, $\varkappa_1 + \varkappa > 30$.

The formula for the first approximation gives for these ratios of sides the values, which differ little from those adduced above.

Panels with parameters of main curvatures, which are smaller than those shown, can be expediently termed weakly-curved plates, but with large values, they should be termed shells, moreover, if in the process of loss of stability on the boundary of the region of instability plastic deformation does not appear, the shell must :e called a sloping shell.

The adduced solution is true for shells with constant Gaussian curvature (cylindrical, spherical and others).

From the solution it is clear that the first approximation gives an exaggerated value of "upper" and decreased value of "lower" critical loads. One may also see that with the increase of the ratio of sides of panel the rigidity of the shell decreases.

Thus, the first two approximations give a practically accurate solution. The analysis of calculations showed that the subsequent third and fourth approximations do not introduce essential corrections either in the magnitude of sags or in the magnitude of critical loads.

Let us examine now the results of M. A. Koltunov's solution on the stability of the panel of a cylindrical rigid sloping shell, subjected to compression along [56]. Approximating functions were selected in the form of the sum of two, three and four terms of the double trigonometric series, satisfying conditions of hinged fastening of edges, Bubnov-Galerkin's procedure was applied and the systems of non-linear algebraic equations obtained were computed by the method of iterations on a "Strela" electronic computer in the Calculation Center of Moscow State University.

Results of calculations are obtained in the form of tables, which are transfered to graphs (load-sag), a portion of which is adduced here.

Let us examine, for instance, the graph in Fig. 42, plotted for cylindrical panels with the ratio of sides $\gamma = 1$ and the sum of main curvatures $\varkappa_1 + \varkappa_2 = \frac{k_1 a^2}{h} + \frac{k_2 b^2}{h}$ equal to 0, 6, 12, 18, 24, 32. Thin lines are used for plotting graphs of the ratio of parameter of sag $x_1 = \frac{f}{h}$ (f is the depth of sag) to the load parameter $r^* = \frac{rb^2}{Eh^2}$ in the

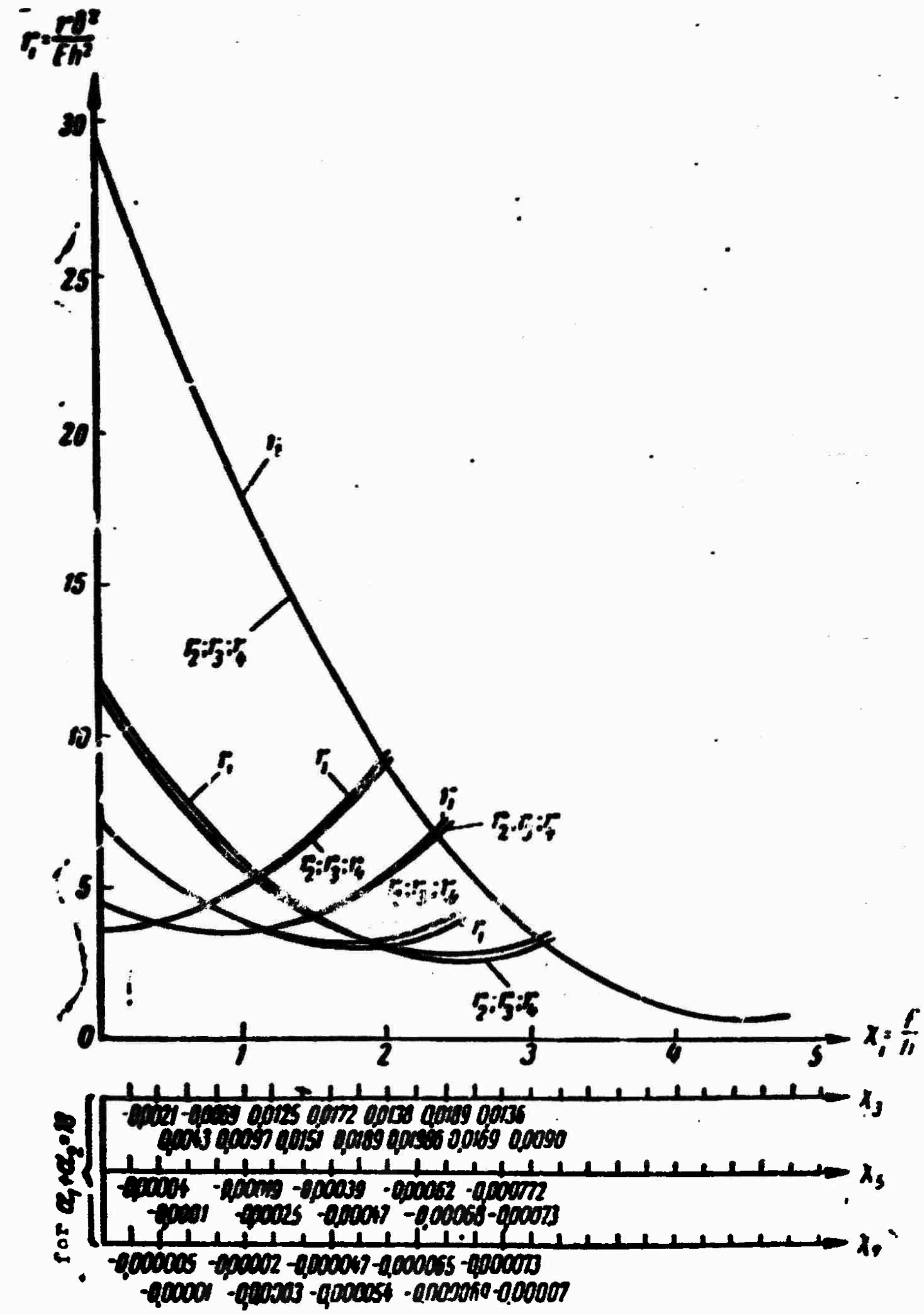



Fig. 42.

solution of the problem in the first approximation, when as approximating for sag w and function of stresses φ we selected one term of the double trigonometric series. The load-sag relationship in this case has the form

$$r(y)\,x_1=\frac{\pi^4\left(\gamma+\frac{1}{\gamma}\right)^2}{12(1-\nu^2)}x+\frac{x_2^2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}x-\frac{16x_2}{\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}x^2+\frac{512}{9\pi^2\left(\gamma+\frac{1}{\gamma}\right)^2}x^3.$$

Here $\varkappa_2 = \frac{k_2 b^2}{h}$ — the parameter of the curvature.

The results of the solution of the problem in higher approximations (two, three and four terms of the series) in view of their small distinction from each other are plotted in a single (thick) line, which prior to values of the lower $r_1$ of the critical stress lies below the curve of the first approximation, and in the stage following buckling lies above the curve.

It is clear that solution of the problem in higher approximations increases the value of the lower critical load; here the distinction between solutions in the fourth approximation and those in the third, and of those of the third from those of the second is insignificant as one may see also from Table 7 given here, where for certain sags ($x_1 = 0.2, 1.0, 2.0$) we give values of parameters of load $r_i$ ($i = 1, 2, 3, 4$), obtained in various approximations, and the difference between them $\Delta_{ij} = \frac{r_j - r_i}{r_j} 100\%$. It is clear that if $\Delta_{12}$ sometimes attains significant values of the order of 50%, then $\Delta_{23}$, $\Delta_{34}$ and $\Delta_{24}$ will not exceed 0.5% anywhere. This indicates the convergence of the Bubnov-Galerkin process for similar nonlinear problems.

Analytical load-sags dependencies for solutions of problems in higher approximations are not given here in view of their cumbersomeness.

It is easier to use graphs and tables. In Fig. 42a and Table 8 we give solutions for cylindrical panels with the side ratio $\gamma = 2$.

We will now examine the results of solution by M. A. Koltunov of the problem on determination of normals, tangents, and intensity of stresses in the middle surface of a flexible sloping shell, using the same selection of approximating functions [57]. Normal and tangential

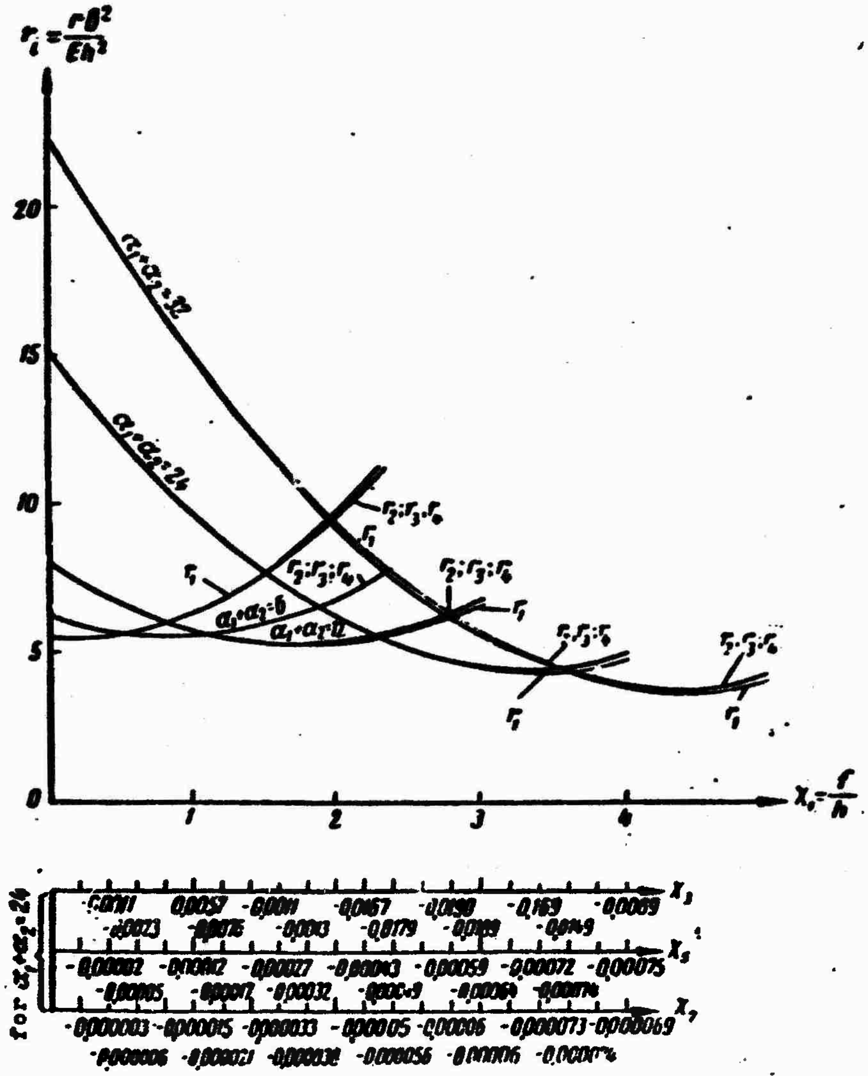


Fig. 42a.

stresses in the middle surface of a flexible sloping shell are calculated by the formulas (1.15) and have the following form:

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} = \sum_m \sum_n \frac{n^2 \pi^2}{b^2} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

$$\tau = -\frac{\partial^2 \varphi}{\partial x \partial y} = -\sum_m \sum_n \frac{mn\pi^2}{ab} A_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$(m = 1,\ 3,\ 5,\ 7; \qquad n = 1,\ 3,\ 5,\ 7).$$

The intensity of stresses is determined by formula

$$\tau_i = \frac{\sqrt{3}}{2} \sqrt{(\sigma_x - \sigma_y)^2 + \tau^2}.$$

Using the former designations, we have:

$$A_{11} = \beta_1 E h^2,$$

Table 7. Values of Load and Error Parameters with Various Approximations to the Solution for $\gamma = 1$.

| $e_1+e_2$ | $x_1$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $\Delta_{12}$ | $\Delta_{22}$ | $\Delta_{32}$ | $\Delta_{42}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0,2 | 3,672 | 3,673 | 3,673 | 3,673 | 0,32% | 0,00% | 0,00% | 0,00% |
| | 1,0 | 5,056 | 5,052 | 5,052 | 5,052 | 0,07% | 0,00% | 0,00% | 0,00% |
| | 2 | 9,379 | 9,252 | 9,251 | 9,251 | 1,35% | 0,10% | 0,00% | 0,10% |
| 6 | 0,2 | 4,097 | 4,098 | 4,098 | 4,098 | 0,02% | 0,00% | 0,00% | 0,00% |
| | 1 | 3,533 | 3,543 | 3,543 | 3,543 | 0,28% | 0,00% | 0,00% | 0,00% |
| | 2 | 5,422 | 5,4094 | 5,4096 | 5,4096 | 0,24% | 0,003% | 0,00% | 0,03% |
| | 0,8 c.l. | 3,5016 | 3,5062 | 3,5062 | 3,5062 | 0,18% | 0,00% | 0,00% | 0,00% |
| 12 | 0,2 | 6,347 | 6,346 | 6,346 | 6,346 | 0,016% | 0,00% | 0,00% | 0,00% |
| | 1 | 3,840 | 3,840 | 3,840 | 3,840 | 0,00% | 0,00% | 0,00% | 0,00% |
| | 2 | 3,300 | 3,329 | 3,330 | 3,330 | 0,87% | 0,03% | 0,00% | 0,03% |
| | c.l. | 3,159 | 3,179 | 3,179 | 3,179 | 0,63% | 0,00% | 0,00% | 0,00% |
| 18 | 0,2 | 10,47 | 10,41 | 10,41 | 10,41 | 0,57% | 0,00% | 0,00% | 0,00% |
| | 1 | 6,190 | 5,941 | 5,941 | 5,941 | 4,02% | 0,00% | 0,00% | 0,00% |
| | 2 | 2,997 | 3,012 | 3,012 | 3,012 | 0,50% | 0,00% | 0,00% | 0,00% |
| | c.l. | 2,586 | 2,642 | 2,642 | 2,642 | 2,16% | 0,00% | 0,00% | 0,00% |
| 32 | 0,2 | 27,01 | 26,98 | 26,97 | 26,97 | 0,11% | 0,037% | 0,00% | 0,037% |
| | 1,0 | 18,01 | 17,61 | 17,61 | 17,61 | 2,22% | 0,00% | 0,00% | 0,00% |
| | 2,0 | 9,349 | 8,835 | 8,836 | 8,836 | 5,50% | 0,01% | 0,00% | 0,01% |
| | c.l. | 0,3183 | 0,7160 | 0,7125 | 0,7125 | 0,96% | 0,56% | 0,00% | 0,56% |

Table 8. Values of Load and Error Parameters with Various Approximations to the Solution for $\gamma = 2$.

| $e_2+e_1$ | $x_1$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $\Delta_{12}$ | $\Delta_{22}$ | $\Delta_{32}$ | $\Delta_{42}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0,2 | 5,643 | 5,643 | 5,643 | 5,643 | 0,00% | 0,00% | 0,00% | 0,00% |
| | 1,0 | 6,571 | 6,572 | 6,572 | 6,572 | 0,015% | 0,00% | 0,00% | 0,00% |
| | 2,0 | 9,348 | 9,312 | 9,312 | 9,312 | 0,38% | 0,00% | 0,00% | 0,00% |
| 6 | 0,2 | 5,958 | 5,958 | 5,958 | 5,958 | 0,00% | 0,00% | 0,00% | 0,00% |
| | 1,0 | 5,598 | 5,602 | 5,602 | 5,602 | 0,07% | 0,00% | 0,00% | 0,00% |
| | 2,0 | 6,808 | 6,813 | 6,813 | 6,813 | 0,07% | 0,00% | 0,00% | 0,00% |
| | c.l. | 5,576 | 5,580 | 5,580 | 5,580 | 0,07% | 0,00% | 0,00% | 0,00% |
| 12 | 0,2 | 7,397 | 7,397 | 7,397 | 7,397 | 0,00% | 0,00% | 0,00% | 0,00% |
| | 1,0 | 5,792 | 5,795 | 5,795 | 5,795 | 0,05% | 0,00% | 0,00% | 0,00% |
| | 2,0 | 5,447 | 5,464 | 5,464 | 5,464 | 0,31% | 0,00% | 0,00% | 0,00% |
| | c.l. | 5,364 | 5,375 | 5,375 | 5,375 | 0,22% | 0,00% | 0,00% | 0,00% |
| 24 | 0,2 | 13,77 | 13,77 | 13,77 | 13,777 | 0,00% | 0,00% | 0,00% | 0,00% |
| | 1,0 | 9,683 | 9,667 | 9,667 | 9,667 | 0,16% | 0,00% | 0,00% | 0,00% |
| | 2,0 | 6,225 | 6,216 | 6,216 | 6,216 | 0,14% | 0,00% | 0,00% | 0.00% |
| | c.l. | 4,482 | 4,545 | 4,545 | 4,545 | 1,40% | 0,00% | 0,00% | 0,00% |
| 32 | 0,2 | 20,62 | 20,62 | 20,62 | 20,62 | 0,00% | 0,00% | 0,00% | 0,00% |
| | 1,0 | 14,87 | 14,82 | 14,82 | 14,82 | 0,33% | 0,00% | 0,00% | 0,00% |
| | 2,0 | 9,337 | 9,256 | 9,256 | 9,256 | 0,87% | 0,00% | 0,00% | 0,00% |
| | c.l. | 3,533 | 3,706 | 3,706 | 3,706 | 4,90% | 0,00% | 0,00% | 0,00% |

where

$$\beta_1 = \frac{4}{\pi^6\left(\gamma + \frac{1}{\gamma}\right)^2}\left(\frac{e_1 + e_2}{4} x_1 - \frac{4}{3} x_1^2\right).$$

In solving the problem in the first approximation we will obtain:

$$\sigma_y^{(1)} = A_{11} \frac{\pi^2}{a^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} =$$
$$= -\frac{Eh^2}{a^2}\left[\frac{1}{\left(\gamma + \frac{1}{\gamma}\right)^2}\left(\frac{e_1 + e_2}{4} x_1 - \frac{4}{3} x_1^2\right)\right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}.$$

In the second approximation –

$$A_{33} = \beta_3 E h^2,$$

where

$$\beta_3 = \frac{4}{81\pi^6\left(\gamma + \frac{1}{\gamma}\right)^2}\left[\frac{9(e_1 + e_2)}{4} x_3 + \frac{4}{5} x_1^2 - \frac{648}{35} x_1 x_3 + 12 x_3^2\right],$$

and the formula for calculation of normal stresses will assume the form

$$\sigma_y^{(2)} = -A_{11} \frac{\pi^2}{a^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} - A_{33} \frac{9\pi^2}{a^2} \sin \frac{3\pi x}{a} \sin \frac{3\pi y}{b} =$$
$$= -\psi\left[\frac{(e_1 + e_2)}{\left(\gamma + \frac{1}{\gamma}\right)^2} x_1 - \frac{16}{3\left(\gamma + \frac{1}{\gamma}\right)^2} x_1^2\right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} -$$
$$- 9\pi^2\psi \frac{4}{81\pi^6\left(\gamma + \frac{1}{\gamma}\right)^2}\left[\frac{9(e_1 + e_2)}{4} x_3 + \frac{4}{5} x_1^2 - \frac{648}{35} x_1 x_3 + \right.$$
$$\left. + 12 x_3^2\right] \sin \frac{3\pi x}{a} \sin \frac{3\pi y}{b}.$$

Here

$$\psi = \frac{Eh^2}{a^2}.$$

Similar formulas will be obtained upon solution of the problem in the third and fourth approximations.

Dependencies of tangents and intensity of stresses on the sag are constructed in an analogous manner. In Fig. 43 we give values

$\frac{\sigma}{\psi}$, $\frac{\tau}{\psi}$ in various points of the square panel of the shell with parameters of curvature $\alpha_1 + \alpha_2 = 24$ with a sag $f = 3.1$ h. Here also we plot curves of equal stresses $\frac{\sigma}{\psi}$ and $\frac{\tau}{\psi}$.

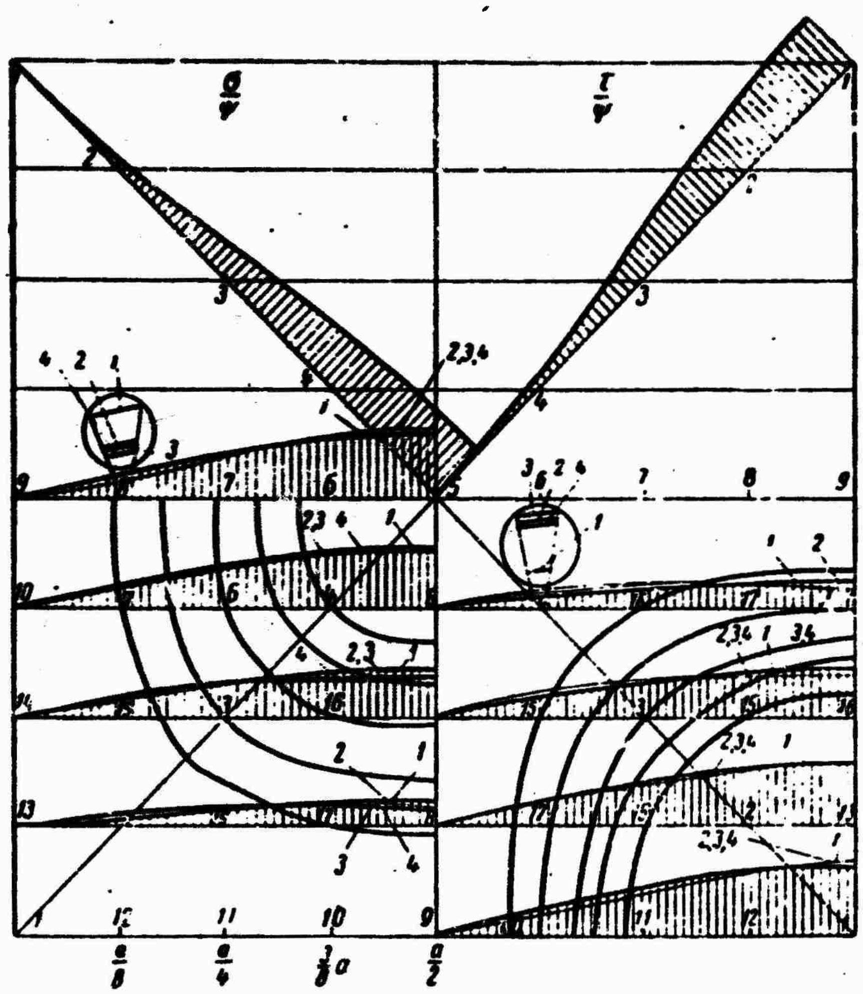


Fig. 43.

From tables of functions $\sigma^{(i)}$, $\tau^{(i)}$, according to which these curves were constructed, it is clear that in the most unfavorable cases the differences in values of stresses, obtained in the first and second approximations are not as big, as they are for the load-sag dependency. The difference between values of stresses, obtained upon solution of the problem in the second and third, third and fourth approximations, as can be seen from tables and graphs given here, is insignificant. With smaller sags this difference is still smaller.

An analysis of curves and corresponding tables for shells with

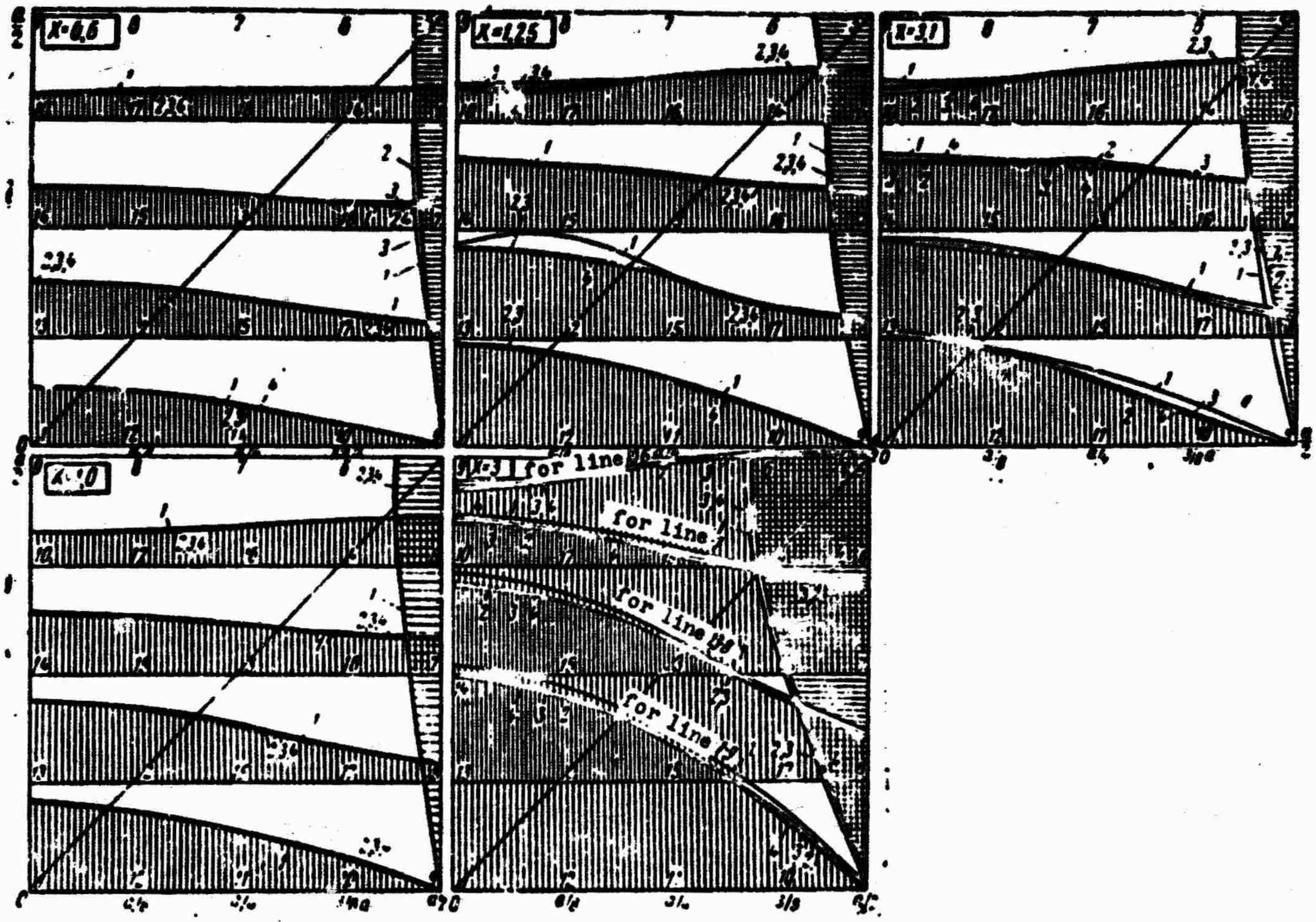



Fig. 44.

curvature parameters from 0 to 60 enables us to assume that the solution of the problem in the second approximation yields sufficiently accurate values of normal and tangential stresses from the middle surface of the panel of a flexible sloping shell, working in the elastic region.

Results of calculations of intensity of stresses in various points of a quarter of the panel with $\gamma = 1.5$; $\alpha_1 + \alpha_2 = 24$ and with various sags $x = 0.6$ h; h; 1.25 h, 3 h, 3.1 h are shown in Fig. 44.

From the graphs it is clear that for determination of the intensity of stresses it is insufficient to be confined to the solution of the problem in the first approximation, since values of stresses at separate sections differ noticeably from values of stresses, obtained upon the solution of the problem in the second approximation. The

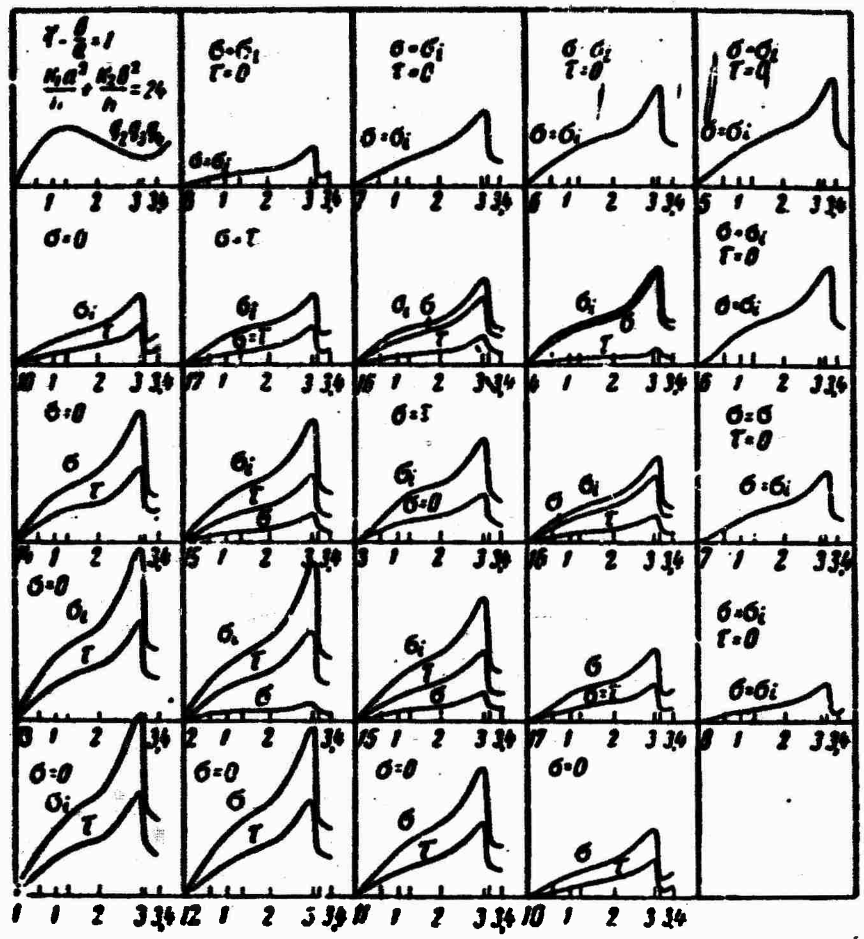


Fig. 45.

difference between stresses of the second and third, third and fourth approximations is insignificant and decreases with the growth of the number of terms of approximating functions.

Let us consider the dependence of stresses $\sigma$, $\tau$, and $\sigma_i$ on sags in various points (1-20) of a quarter of the panel (in the other three quarters the picture will be analogous). In Fig. 45 we give graphs of such dependencies. For instance, at point 15 with coordinates $x = \frac{a}{8}$, $y = \frac{a}{4}$ curves are plotted for the dependence $\sigma$, $\tau$, and $\sigma_i$ on the sag of this point.

From the examination of curves in various points it follows that the biggest values of stress in these points are attained with the lower critical value of load.

Let us note that for calculation of stresses in the critical

state of the shell it is convenient to use the above-mentioned formulas for stresses, where it is required to introduce values of sag parameters x, taken from graphs of this type in Fig. 41.

Let us note that a large quantity of works on the dynamic stability of flexible shell panels is solved by the Bubnov-Galerkin method. In these solutions the force of inertia and damping is introduced in supplement to the external load and the same operations are carried out, as for instance, those which are given here with the subsequent analysis of solutions. Therefore, without dwelling on these questions in detail, let us give here one of the solutions on the dynamic stability of the sloping cylindrical shell, given by G. V. Mishenkov [58].

## § 3. Dynamic Stability of the Sloping Shell Panel

The problem of dynamic stability of thin-walled elements and structures as a whole is of great interest for technology. It is especially important to know the character and magnitude of loads, at which displacement of points of a material system (body) begin to increase in time without a limit which can produce in the beginning a disturbance of given working conditions and later results in destruction.

For the first time the problem of dynamic stability in reference to elastic rods was considered by N. M. Belyayev [59] in 1924. A quarter of a century later M. A. Lavrent'yev and A. Yu. Ishlinskiy [60] investigated phenomena, caused by the action of shock or sudden application of loads on the rod; they showed that upon a sudden application of the load, exceeding n-th critical static force, the appearance is possible of the n-th stable form of equilibrium, which has n half-waves. This result was verified experimentally by its authors. A series of works of foreign authors (61, 62), on the research of rod stability is known.

In 1938-1939 V. N. Chelomey [63] considered a number of problems on the dynamic stability of aeronautical structures. A number of investigations of this kind of theoretical and experimental character was carried out by other authors [64-67].

In 1955 A. I. Blokhina [67] solved the problem on the dynamic stability of the cylindrical shell, supported with hinge, edges. We know of the research of a number of other authors both Soviet and foreign. A sufficiently complete survey of contemporary trends in the field of dynamics of plates and shells and a large bibliography are given by V. V. Bolotin; those interested should refer to his

article.* Below we expound the solution, obtained by G. V. Mishenkov [58].

Let us consider an elastic cylindrical shell with radius R, resting on a rectangular contour with sides a and b. It is assumed that sags of the shell are comparable with its thickness, but are sufficiently small as compared to other dimensions of the shell. It is also assumed that the natural frequencies of tangential oscillations are sufficiently large in comparison with the frequency of external action. This allows us to disregard tangential inertia.

Taking into account the assumptions given the deformation of the shell is described by the system of nonlinear equations,

$$\begin{gathered} D\nabla^2\nabla^2 w = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \varphi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 \varphi}{\partial x\,\partial y} + \frac{1}{R}\frac{\partial^2 \varphi}{\partial x^2} - q, \\ \frac{1}{Eh}\nabla^2\nabla^2\varphi = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}. \end{gathered} \tag{3.1}$$

For the oscillative shell the normal load is calculated as the sum of forces of inertia, damping and external load,

$$q(x, y, t) = -\rho_0 h\frac{\partial^2 w}{\partial t^2} - 2\rho_0 h\varepsilon\frac{\partial w}{\partial t} + q_0(x, y, t), \tag{3.2}$$

where $\rho_0$ is the density of the shell's material, $\varepsilon$ — characteristic of damping, $q_0(x, y, t)$ — external load.

Let us assume, that the shell examined is supported along the contour, but at the ends is loaded by axial periodic forces $p = p_0 + p_t \cos \theta t$, distributed evenly on the generator of the middle surface. Let us also assume that normal forces, acting on the longitudinal edges $y = 0$, $y = b$, "on the average" are equal to zero. Boundary

*Transactions II of the All-Union Conference on the Theory of Plates and Shells. Kiev, 1962.

conditions can be written in the form,

$$
\begin{gathered}
w(0, y) = w(a, y) = w(x, 0) = w(x, b) = 0, \\
\frac{1}{b}\int_0^b \frac{\partial^2 \varphi(0, y)}{\partial y^2}\,dy = -(p_0 + p_t \cos\theta t), \\
\frac{1}{a}\int_0^a \frac{\partial^2 \varphi(x, 0)}{\partial x^2}\,dx = 0.
\end{gathered}
\tag{3.3}
$$

Let us seek the solution in the form of series

$$
w(x, y, t) = \sum_{i,j=1}^{\infty} \xi_{ij}(t) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}, \tag{3.4}
$$

where $\xi_{ij}(t)$ are as yet unknown time function. Since we will investigate resonance phenomena, connected with the main form of oscillation, then we will take only the first term in the series:

$$
w(x, y, t) = \xi(t) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{3.5}
$$

Placing this in the second equation of the system of equations (3.1), we write its solution in the form,

$$
\begin{gathered}
\varphi(x, y, t) = \frac{Eh}{32}\,\xi^2(t)\left[m^2\cos\frac{2\pi x}{a} + \frac{1}{m^2}\cos\frac{2\pi y}{b}\right] + \\
+ \frac{Eh}{R\pi^2}\,\frac{a^2\xi(t)}{(1+m^2)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{2}p_y^0 x^2 + \frac{1}{2}p_x^0 y^2 - p_{xy}^0 xy,
\end{gathered}
\tag{3.6}
$$

where $m = a/b$.

Parameters $p_x^0$ and $p_y^0$ are determined from the last boundary conditions, and parameter $p_{xy}^0$, characterizing tangent forces on edges, is assumed to equal zero. We put the expression for the given function and (3.5) in the first equation of the system of equations (3.1), by the Bubnov-Galerkin method and obtain the ordinary differential equation with periodic coefficients with respect to $\xi(t)$:

$$
\begin{gathered}
\frac{d^2\xi}{dt^2} + 2\varepsilon\frac{d\xi}{dt} + \omega^2\xi - \xi\,\frac{\pi^2}{a^2}(p_0 + p_t\cos\theta t) - \xi^2\,\frac{Eh}{3Rb^2}\left[\frac{1}{2} + \frac{8}{(1+m^2)^2}\right] + \\
+ \xi^3\,\frac{Eh}{16}\,\frac{\pi^4}{a^4}(1 + m^4) = 0.
\end{gathered}
\tag{3.7}
$$

Here $\omega$ is the natural frequency of transverse oscillations, determined by the formula

$$\omega^2 = \frac{\pi^4}{a^4}\frac{D}{\rho h}F(m, k), \tag{3.8}$$

where through F(m, k) we designated the geometric characteristic of the examined shell

$$F(m, k) = (1 + m^2)^2 + \frac{12(1-\nu^2)k^2}{\pi^4(1+m^2)^2}; \qquad k = \frac{a^2}{Rh}. \tag{3.9}$$

We reduce the equation (3.7) to another form. For that we introduce values of critical parameter $p_*$ for static loading of the shell by forces distributed longitudinally

$$p_* = \frac{\pi^2 D}{a^2}F(m, k), \tag{3.10}$$

of the coefficient of excitation

$$\mu = \frac{p_t}{2(p_* - p_0)} \tag{3.11}$$

and natural frequency, taking into account loading of shell by the constant component of longitudinal force

$$\Omega^2 = \omega^2\left(1 - \frac{p_0}{p_*}\right). \tag{3.12}$$

If we also change to dimensionless amplitude and

$$\zeta(t) = \frac{\xi(t)}{h},$$

and designate through $\alpha$ and $\beta$ the coefficients, characterizing the geometric nonlinearity

$$\alpha = \frac{3}{4}\frac{1-\nu^2}{F(m, k)}(1 + m^4), \qquad \beta = \frac{16(1-\nu^2)km^2}{\pi^4 F(m, k)}\left[\frac{1}{2} + \frac{8}{(1+m^2)^2}\right], \tag{3.13}$$

then the equation (3.7) will be written in the form

$$\frac{d^2\zeta}{dt^2} + 2\varepsilon\frac{d\zeta}{dt} + \Omega^2[\zeta - 2\mu\zeta\cos\theta t + \bar{\alpha}\zeta^2 - \bar{\beta}\zeta^3] = 0. \tag{3.14}$$

where

$$\bar{\varepsilon} = \frac{\varepsilon}{1 - \frac{P_0}{P_*}}, \qquad \bar{\beta} = \frac{\beta}{1 - \frac{P_0}{P_*}}. \tag{3.15}$$

Here and in the future the line above $\xi(t)$ is omitted. Equation (3.14) differs from the analogous equation for the plate by the presence of a quadratic term, characterizing the asymmetric character of nonlinearity, inherent in shells. It should be considered as the first approximation for description of parametrically excited oscillations in shells.

A further more precise determination is possible in examining a large number of terms of the approximating series, resulting in a system of equations of a similar type. As we know (14), the instability region, of zero solution (3.14) lie near frequencies

$$\theta = \frac{2\Omega}{k} \quad (k = 1, 2, \ldots). \tag{3.16}$$

The most dangerous is first (main) region of instability, the boundaries of which damping calculations excluded are determined by the approximate formula

$$\frac{\theta}{\Omega} = 2\sqrt{1 \mp \mu}. \tag{3.17}$$

Let us search for the periodic solutions of equation (3.14) in the neighborhood of the main region of instability, disregarding damping. If as the first approximation we would be limited by considering the solution in the form

$$\xi(t) = b_1 \cos\frac{\theta t}{2} \tag{3.18}$$

for the branch, adjoining the lower boundary of the region of instability, and

$$\xi(t) = a_1 \sin\frac{\theta t}{2} \tag{3.19}$$

for the other branch, then for the amplitude of steady-state oscillations e obtain formulas:

$$b_1^2 = \frac{4}{3}\bar{\varepsilon}\sqrt{\frac{n^4}{4} - 1 - \mu}. \tag{3.20}$$

$$a_1^2 = \frac{4}{3}\bar{\varepsilon}\sqrt{\frac{n^4}{4} - 1 + \mu}. \tag{3.21}$$

where $n = \theta/\Omega$. These formulas coincide with corresponding formulas for plates [14]. Meanwhile it is natural to expect qualitative distinction in the behavior of shells during parametric oscillations. Apparently, for shells it is insufficient to be limited by harmonious approximation (3.18) and (3.19). In this approximation the quadratic term in equation (3.14), which reflects the specific character of nonlinearity of the shell, is not considered.

From combinational considerations it ensues that the second approximation should be sought in the form

$$\xi(t) = b_0 + b_1\cos\frac{\theta t}{2} + b_2\cos\theta t \tag{3.22}$$

for the branch, originatives from the lower boundary of main instability, and in the form

$$\xi(t) = a_0 + a_1\sin\frac{\theta t}{2} + a_2\cos\theta t \tag{3.23}$$

for the other branch. Terms, containing $\cos\theta t$, are added in order to take into consideration the deflection of solutions from purely harmonious solutions of the first approximation. The inclusion in solution of the free term is natural. Actually, if we place $\cos\theta t$ in nonlinear part of (3.14), then the result of substitution will contain the constant term, having the same order as the coefficient of $\cos\theta t$. We place solution (3.22) in equation (3.14) and equate to zero coefficients of the absolute term, $\cos\frac{\theta t}{2}$ and $\cos\theta t$. The system of nonlinear algebraic equations thus produced is too bulky

for research and is not adduced here. It can be significantly simplified if we take into consideration that in solution (3.22) the term constaining $\cos\frac{\theta t}{2}$ is the determining term. Proceeding from this, we consider that $b_1 \gg b_0$, $b_1 \gg b_2$ which makes it possible to disregard degrees higher than the first and the product of values $b_0$ and $b_2$. Then the system of algebraic equations will be written in the form

$$\begin{gathered}
b_0-\mu b_2+\frac{3}{2}\bar{\alpha}b_1^2\left(b_0+\frac{b_2}{2}\right)-\frac{1}{2}\bar{\beta}b_1^2=0,\\
b_2-2\mu b_0-n^2b_2+\frac{3}{2}\bar{\alpha}b_1^2(b_0+b_2)-\frac{1}{2}\bar{\beta}b_1^2=0,\\
1-\frac{n^2}{4}-\mu+\frac{3}{4}\bar{\alpha}b_1^2-\bar{\beta}(2b_0+b_2)=0.
\end{gathered}\tag{3.24}$$

Excluding from this system of equations parameters $b_0$ and $b_2$, we obtain the following equation for $b_1$,

$$\begin{gathered}
\frac{27}{32}\bar{\alpha}^2b_1^6+\left(\frac{27}{8}\bar{\alpha}^2+\frac{9}{8}\bar{\alpha}^2\mu-\frac{3}{4}\bar{\alpha}\bar{\beta}^2-\frac{45}{32}\bar{\alpha}^2n^2\right)b_1^4+\\
+\left[\frac{15}{4}\bar{\alpha}-\frac{9}{2}\bar{\alpha}\mu^2-\frac{3}{2}\bar{\beta}^2-2\bar{\beta}^2\mu+n^2\left(\frac{3}{4}\bar{\alpha}\mu+\bar{\beta}^2-3\bar{\alpha}\right)+\right.\\
\left.+\frac{3}{8}\bar{\alpha}n^4\right]b_1^2+\frac{n^4}{4}-n^2\left(\frac{5}{4}-\frac{\mu^2}{2}-\mu\right)+1-\mu-2\mu^2+2\mu^3=0.
\end{gathered}\tag{3.25}$$

For solution of (3.23) the system of algebraic equations has the form,

$$\begin{gathered}
a_0-\mu a_2+\frac{3}{2}\bar{\alpha}a_1^2\left(a_0-\frac{1}{2}a_2\right)-\frac{1}{2}\bar{\beta}a_1^2=0,\\
a_2-2\mu a_0-n^2a_2+\frac{3}{2}\bar{\alpha}a_1^2(a_2-a_0)+\frac{1}{2}\bar{\beta}a_1^2=0,\\
1+\mu-\frac{n^2}{4}+\frac{3}{4}\bar{\alpha}a_1^2-\bar{\beta}(2a_0-a_2)=0.
\end{gathered}\tag{3.26}$$

For amplitude $a_1$ analogously we obtain,

$$\begin{gathered}
\frac{27}{32}\bar{\alpha}^2a_1^6+\left(\frac{27}{8}\bar{\alpha}^2-\frac{9}{8}\bar{\alpha}^2\mu-\frac{3}{4}\bar{\alpha}\bar{\beta}^2-\frac{45}{32}\bar{\alpha}^2n^2\right)a_1^4+\\
+\left[\frac{15}{4}\bar{\alpha}-\frac{9}{2}\bar{\alpha}\mu^2-\frac{3}{2}\bar{\beta}^2+2\bar{\beta}^2\mu+n^2\left(-\frac{3}{4}\bar{\alpha}\mu+\bar{\beta}^2-3\bar{\alpha}\right)+\right.\\
\left.-\frac{3}{8}\bar{\alpha}n^4\right]a_1^2+\frac{n^4}{4}-n^2\left(\frac{5}{4}-\frac{\mu^2}{2}+\mu\right)+1+\mu-2\mu^2-2\mu^3=0.
\end{gathered}\tag{3.27}$$

As we should have expected, the coefficient of the quadratic term is

included in the equation for amplitudes (3.25) and (3.27).

Further more precise determination can be carried out, presenting the solution for corresponding branches in the form,

$$
\begin{aligned}
\xi(t) &= b_0 + \sum_{i=1,3,\ldots}^{\infty} b_i \cos\frac{i\theta t}{2} + \sum_{k=2,4,\ldots}^{\infty} b_k \cos\frac{k\theta t}{2}, \\
\xi(t) &= a_0 + \sum_{i=1,3,\ldots}^{\infty} a_i \sin\frac{i\theta t}{2} + \sum_{k=2,4,\ldots}^{\infty} a_k \cos\frac{k\theta t}{2}.
\end{aligned}
\tag{3.28}
$$

The system of nonlinear algebraic equations obtained thereby can be solved by approximate methods.

Let us note that in solving equation (3.14) by the method of small parameter the first approximation coincides with formulas (3.20) and (3.21) for $b_1^2$ and $a_1^2$. However the construction of subsequent approximations is very difficult, since as generating equation it is necessary to consider the equation of the Lyapunov type.

Equations (3.25) and (3.27) are easily solved with respect to n, if here the sought amplitude is considered to be a parameter.

The stability of solutions obtained is investigated by known methods [14]. The solution, originating at the lower boundary of the region of instability, is stable, if $\frac{db_1}{dn} > 0$, and unstable, if $\frac{db_1}{dn} < 0$. Another solution, which in the case of the plate, is on the whole unstable, will be stable, if $\frac{da_1}{dn} < 0$. Let us assume, that the frequency of external load increases gradually, passing through regions of instability. Then on the lower boundary of the region of instability we will observe "hard" excitation of steady-state oscillations. Upon a reverse change of frequencies on the upper boundary of the region of instability we will have "soft" excitation.

Calculations [59] performed show that the solution essentially differs from the harmonic solution.

## § 4. The Circular Cylindrical Shell

Let us consider a closed circular cylindrical shell, subjected to the action of evenly distributed external pressure.* We will consider that by the hinged shell ends are fastened with rigid frames; frames can be deformed in their own plane, remaining circular. We assume that the middle surface of the shell has a certain initial bend.

Let us investigate the behavior of the shell upon the increase of the load, considering initial and additional sags comparable with the thickness of the shell. Let us proceed from nonlinear equations of the theory of flexible shells taking into account the initial incorrectnesses in the shape of the middle surface,

$$\frac{D}{h}\nabla^2\nabla^2 w = \frac{\partial^2 (w + w_{\text{нч}})}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 (w + w_{\text{нч}})}{\partial y^2}\frac{\partial^2 \varphi}{\partial x^2} - 2\frac{\partial^2 (w + w_{\text{нч}})}{\partial x \partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{1}{R}\frac{\partial^2 \varphi}{\partial x^2} + \frac{q}{h}, \tag{4.1}$$

$$\frac{1}{E}\nabla^2\nabla^2 \varphi = \left[\frac{\partial^2 (w + w_{\text{нч}})}{\partial x \partial y}\right]^2 - \frac{\partial^2 (w + w_{\text{нч}})}{\partial x^2}\frac{\partial^2 (w + w_{\text{нч}})}{\partial y^2} - \left[\left(\frac{\partial^2 (w_{\text{нч}})}{\partial x \partial y}\right)^2 - \frac{\partial^2 (w_{\text{нч}})}{\partial x^2}\frac{\partial^2 (w_{\text{нч}})}{\partial y^2}\right] - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}, \tag{4.2}$$

[НЧ = in = initial]

where w and $w_{in}$ are the additional and initial sags, φ is the function of stresses in the middle surface, h — thickness, R — radius of the middle surface and L — length of shell.

Coordinates x and y are counted off along the generator and along the arc (Fig. 46).

In selecting approximating expressions for sag we assume that the shape and location of initial dents correspond to the shape and location of dents, formed in the process of deformation.

---

*The solution of problem belongs V. Ye. Mineyev [69].

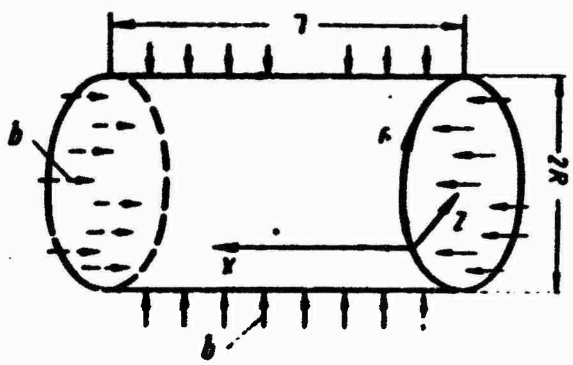



Fig. 46.

The expression for the function of initial sag will be taken in the form

$$w_{n4} = f_{n4}(\sin \alpha x \sin \beta y + \psi \sin^2 \alpha x + \chi). \qquad (4.3)$$

for additional sag – in the form

$$w = f(\sin \alpha x \cdot \sin \beta y + \psi \sin^2 \alpha x + \chi). \qquad (4.4)$$

Full sag is equal to

$$w_n = w + w_{n4}. \qquad (4.5)$$

We assume that

$$\alpha = \frac{\pi}{L},\ \beta = \frac{n}{R}. \qquad (4.6)$$

The expression selected for additional sag (4.4) does not satisfy the condition of hinged support of the shell butts $M_x = 0$ when $x = 0$. This circumstance, however, should not show in the results of the solution of the problem with the selected parameters of the shells.

The first term of expression (4.4) satisfies the form of wave formation of the shell during the "minor" loss of stability. The second term accounts for the preferential deformation of shell toward the center of curvature. The third member characterizes radial deformation of butt frames; it is assumed that butt sections receive radial pressing not only up to the loss of stability, but also in the process of deformation of the shell.

We substitute in the right part of equation (4.2) expressions (4.3), (4.4) and (4.5); integrating it, we shall obtain the following expression for the stress function,

$$\frac{1}{E}\varphi = r_1 \cos 2\alpha x + r_2 \cos 2\beta y + r_3 \sin \alpha x \cdot \sin \beta y + r_4 \sin 3\alpha x \cdot \sin \beta y - \frac{p_1 y^2}{2E\delta} - \frac{p_2 x^2}{E\delta}. \qquad (4.7)$$

Here we introduce designation,

$$r_1 = \frac{s}{32v^4} - \frac{f}{8R}\frac{\psi}{\beta^2 v^4}, \quad r_2 = \frac{1}{32}v^2 s,$$
$$r_3 = \frac{v^2}{(1+v^2)^2}\left(\frac{f}{\beta^2 R} - s\psi\right), \quad r_4 = \frac{v^2}{(1+9v^2)^2}s\psi,$$
$$S = (2f_{m} + f)f. \quad v = \frac{\alpha}{\beta} \qquad (4.8)$$

The terms $p_1$ and $p_2$ designate mean values of compressing forces, per unit of length of annular and longitudinal sections of the shell. They are equal respectively, to

$$p_1 = \frac{qR}{2}, \quad p_2 = qR. \qquad (4.9)$$

The strain energy of the middle surface of the shell is determined by expression

$$U_1 = \frac{h}{2E}\int_0^L\int_0^{2\pi R}\left\{\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)^2 - 2(1+\nu)\times\right.$$
$$\left.\times\left[\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} - \left(\frac{\partial^2\varphi}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy. \qquad (4.10)$$

Let us place in equation (4.10) expression (4.7); after integration we obtain;

$$U_1 = \frac{\pi REhL}{2}\left\{\frac{\beta^4(1+v^4)}{64}s^2 + \frac{1}{2}\beta^4\left[\frac{v^4}{(1+v^2)^2} + \frac{v}{(1+9v^2)^2}\right]s^2\psi^2 - \right.$$
$$-\frac{1}{8}\frac{\beta^2}{R}\left[1 + \frac{8\alpha^4}{(1+v^2)^2}\right]sf\psi + \frac{1}{2}\frac{1}{R^2}\frac{v^4}{(1+v^2)^2}f^2 + \frac{1}{4}\frac{1}{R^2}f^2\psi^2 +$$
$$\left.+\frac{2}{E^2h^2}(p_1^2 + p_2^2 - 2\mu p_1 p_2)\right\}. \qquad (4.11)$$

The strain energy of bend $U_2$ is equal

$$U_2 = \frac{D}{2}\int_0^L\int_0^{2\pi R}\left\{\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \right.\right.$$
$$\left.\left.-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy. \qquad (4.12)$$

Placing in expression (4.12) expression (4.4) and integrating,

we will have

$$U_2 = \frac{\pi R L E h^3}{24(1-\nu^2)}\left[\frac{1}{2}\beta^4(1+v^2)^2 f^2 + 4v^4\beta^4 f^2\psi^2\right]. \qquad (4.13)$$

The work of external transverse pressure is determined by expression

$$W_2 = p_2\int_0^L\int_0^{2\pi R} w\,dx\,dy. \qquad (4.14)$$

After substitution and integration we will find,

$$W_2 = 2\pi L p_2 f\left(\chi + \frac{1}{2}\psi\right). \qquad (4.15)$$

The work of compressing forces along the generator of the shell is equal to

$$W_1 = \frac{p_1}{E}\int_0^L\int_0^{2\pi R}\left\{\left(\frac{\partial^2\varphi}{\partial x^2} - \nu\frac{\partial^2\varphi}{\partial y^2}\right) - \frac{E}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \right.$$

$$\left. - \frac{E}{2}\left(\frac{\partial w}{\partial x}\frac{\partial w_{\text{н}}}{\partial x}\right)\right\}dx\,dy. \qquad (4.16)$$

We place expressions (4.3), (4.4) and (4.7) in equation (4.16) and, integrating, will obtain,

$$W_1 = \frac{\pi R L}{2Eh}\left\{4p_1^2 - \nu 2p_1^2 + Ep_1 h\left[\frac{1}{2}L^2 f(f + 2f_{\text{н}}) + L^2 f(f + 2f_{\text{н}})\right]\right\}. \qquad (4.17)$$

We replace in expressions (4.11), (4.13) and (4.17) $p_1$ and $p_2$ with q according to (4.9).

Further, we use condition of closure

$$\int\frac{\partial v}{\partial y}dy = 0, \qquad (4.18)$$

where v is the displacement of along arc.

We set up the expression

$$\frac{\partial v}{\partial y} = \frac{1}{E}\left[\frac{\partial^2\varphi}{\partial x^2} - \nu\frac{\partial^2\varphi}{\partial y^2}\right] - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{\partial w}{\partial y}\frac{\partial w_{\text{н}}}{\partial y} + \frac{1}{R}w. \qquad (4.19)$$

After substitution, integration and rejection of periodic terms will give the equation of closure the form

$$\frac{p_2}{2Eh}(2+\nu) - \frac{1}{8}\beta^2 s + \frac{f}{R}\left(\chi + \frac{1}{2}\psi\right) = 0. \qquad (4.20)$$

We place expression (4.20) in equation (4.15), first replacing in it $p_2$ by q, then we will have

$$W_2 = \frac{\pi R L E h}{2}\left[\frac{1}{2}\frac{R}{Eh}q\beta^2 s - 2\frac{R^2}{Eh^2}q^2(2-\nu)\right]. \tag{4.21}$$

Let us introduce dimensionless parameters for energy, load, and sag,

$$\begin{gathered} \bar{U}_i = U_i\frac{R}{\pi E L h^3},\quad \bar{W}_i = W_i\frac{R}{\pi E L h^3},\quad \hat{q} = \frac{q}{E}\left(\frac{R}{h}\right)^2, \\ \xi = \frac{f}{h},\quad \xi_{нач} = \frac{f_{нач}}{h},\quad \zeta = \frac{f_2}{h},\quad \lambda = \frac{f_1}{h}. \end{gathered} \tag{4.22}$$

Let us set up the expression for total energy of the system

$$\text{Э} = U_1 + U_2 - W_1 - W_2 \tag{4.23}$$

and use the dimensionless parameter

$$\bar{\text{Э}} = \text{Э}\frac{R}{\pi E L h^3},$$

for which we obtain expression

$$\begin{gathered} \bar{\text{Э}} = \frac{c_1}{2}\xi^2(\xi + 2\xi_{нач})^2 + \frac{c_2}{2}(\xi + 2\xi_{нач})^2\zeta^2 - c_3(\xi + 2\xi_{нач})\xi\zeta + \\ + \frac{c_4}{2}\xi^2 + \frac{c_5}{2}\zeta^2 - c_6\hat{q}\xi(\xi + 2\xi_{нач}) - \frac{c_7}{4}\hat{q}\xi(\xi + 2\xi_{нач}) - \\ - \frac{c_7}{2}\hat{q}\frac{\zeta^2}{\xi}(\xi + 2\xi_{нач}) + \frac{c_8}{2}\hat{q}^2, \end{gathered} \tag{4.24}$$

where

$$\begin{gathered} c_1 = \frac{(1+\vartheta^4)}{64}\eta_i^2,\quad c_2 = \frac{1}{2}\left[\frac{\vartheta^4}{(1+\vartheta^2)^2} + \frac{\vartheta^4}{(1+9\vartheta^2)^2}\right]\eta_i^2, \\ c_3 = \frac{1}{16}\left[1 + \frac{8\vartheta^4}{(1+\vartheta^2)^2}\right]\eta_i^2, c_4 = \frac{1}{2}\left[\frac{\vartheta^4}{(1+\vartheta^2)^2} + \frac{(1+\vartheta^2)\eta_i^2}{12(1-\nu^2)}\right], \\ c_5 = \frac{1}{4} + \frac{\vartheta^4}{3(1-\nu^2)}\eta_i^2,\quad c_6 = \frac{1}{4}\eta,\quad c_7 = \frac{1}{2}\varkappa, \\ c_8 = 1 + 2\nu,\quad \eta = \beta^2 Rh,\quad \varkappa = L^2 Rh. \end{gathered} \tag{4.25}$$

Ritz method equations in application to parameters ξ and ζ will be written in the form,

$$\begin{gathered} \frac{\partial \bar{\text{Э}}}{\partial \xi} = 0, \\ \frac{\partial \bar{\text{Э}}}{\partial \zeta} = 0. \end{gathered} \tag{4.26}$$

Upon performing necessary calculations, we obtain,

$$\frac{\partial Э}{\partial \xi} = 2c_1\xi(\xi + 2\xi_{in})(\xi + \xi_{in}) + c_2(\xi + 2\xi_{in})\zeta^2 - - 2c_3(\xi + \xi_{in})\zeta + c_4\xi - 2c_5\hat{q}(\xi + \xi_{in}) - - \frac{1}{2}c_6\hat{q}(\xi + \xi_{in}) = 0; \tag{4.27}$$

$$\frac{\partial Э}{\partial \zeta} = c_2(\xi + 2\xi_{in})\zeta - - c_3\xi(\xi + 2\xi_{in}) + c_6\zeta - - c_7\hat{q}\frac{\zeta}{\xi}(\xi + \xi_{in}) = 0. \tag{4.28}$$

In equation (4.28) we disregard term $c_7\hat{q}\,\frac{\zeta}{\xi}\,\xi_{in}$, and in equation (4.27) – coefficient 2 with $\xi_{in}$ in the last component. The assumptions taken are insignificant and have a small effect on the accuracy of the solution.

Thus, solving systems (4.27), (4.28), we obtain in the final form two equations, connecting the value $\hat{q}$ with parameters $\xi_{in}$, $\xi$, $\zeta$ and the number of waves n along the arc of shell,

$$\hat{q} = \frac{c_1}{N}\xi(\xi + 2\xi_{in}) + \frac{c_2}{2N}\frac{\xi + 2\xi_{in}}{\xi + \xi_{in}}\zeta^2 - \frac{c_3}{N}\zeta + + \frac{c_4}{2N}\frac{\xi}{\xi + \xi_{in}}; \tag{4.29}$$

$$c_2c_7(\xi + 2\xi_{in})\zeta^3 - 2c_2c_7(\xi + \xi_{in})\zeta^2 + [2c_1c_7\xi(\xi + \xi_{in})(\xi + + 2\xi_{in}) - 2c_2N\xi(\xi + 2\xi_{in})^2 - 2c_6N\xi + c_6c_7\xi]\zeta + + 2c_3N\xi^2(\xi + 2\xi_{in}) = 0. \tag{4.30}$$

If in equation (4.29) we put $\xi_{in} = \zeta = 0$ and take only linear terms with respect to $\xi$, we will obtain the upper critical load,

$$\hat{q}^{\text{в}}_{\text{кр}} = \frac{1}{1 + \frac{1}{2}\vartheta^2}\left[\frac{\vartheta^4}{(1 + \vartheta^2)^2\gamma_1} + \frac{(1 + \vartheta^2)^2\gamma_1}{12(1 + \nu^2)}\right]. \tag{4.31}$$

This form was for the first time obtained by Von Mises.

If however, in initial relationships we assume $p_1 = 0$, then we come to dependencies, obtained by A. S. Vol'mir in work [66] for the case of action of only one transverse pressure.

Let us adduce the results of calculations, performed proceeding from full nonlinear equations (4.29) and (4.30) for the case of smooth shell $\xi_{in} = 0$ and shells with initial dents of different magnitude $\xi_{in}$, which changed from 0.001 to 2.0. For the purpose of comparison of the obtained results with the experimental data in addition to ratios R/h = 200, 500 and 1000 the ratios R/h = 112.5 and L/R = 2.45 were also selected.

In calculations the value $\zeta$ was determined by the approximate formula

$$\zeta = \frac{c_4 \xi^2}{c_5 \xi^2 + c_6},$$

and then was made more accurate with cubic equation (4.30).

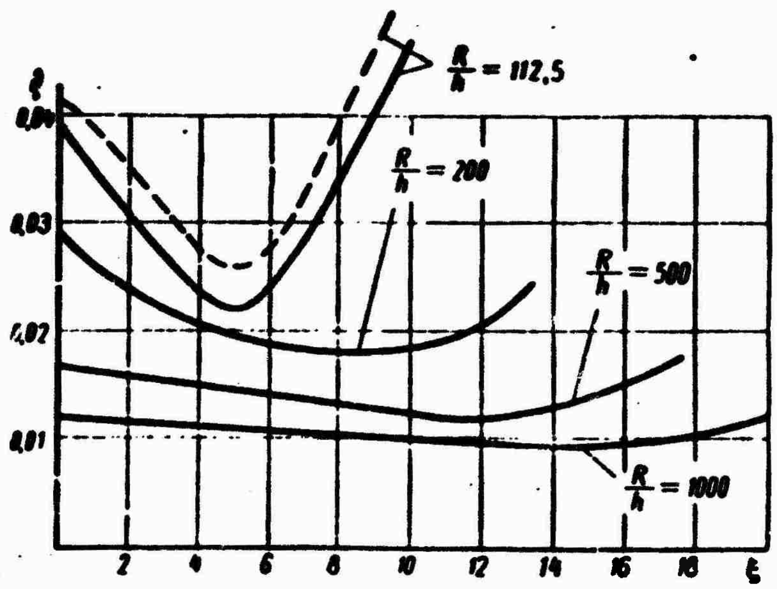



Fig. 47.

In Fig. 47 we represent envelope families of curves for different values of R/h, plotted with equations (4.29) and (4.30), for a smooth shell. The dotted curve pertains to the case of one transverse pressure. We see that the lower critical pressure $q^1_{cr} = 0.022$ for shell R/h = 112.5 for hydrostatic pressure turned out to be 15% less

than for the case of one transverse pressure. In V. E. Mineyev's experiment the critical load for the smooth shell was obtained close to the upper critical value $q^u_{cr} = 0.039$ satisfying formula (4.31). It is also necessary to note that with the increase of relationship the critical value of pressure decreases, and the point, corresponding to the minimum of curve — to the lower critical pressure, is displaced to the right along the $\xi$ axis. It is necessary to consider also the circumstance that with the growth of relationship R/h the critical value of numbers of waves $n_{cr}$ increases thus, for R/h = 112.5 $n_{cr} = 6$, and for R/h = 1,000 $n_{cr} = 10$.

Calculations, performed for shells with initial incorrectnesses in the shape of the middle surface, are represented in Figs. 48 and 49.

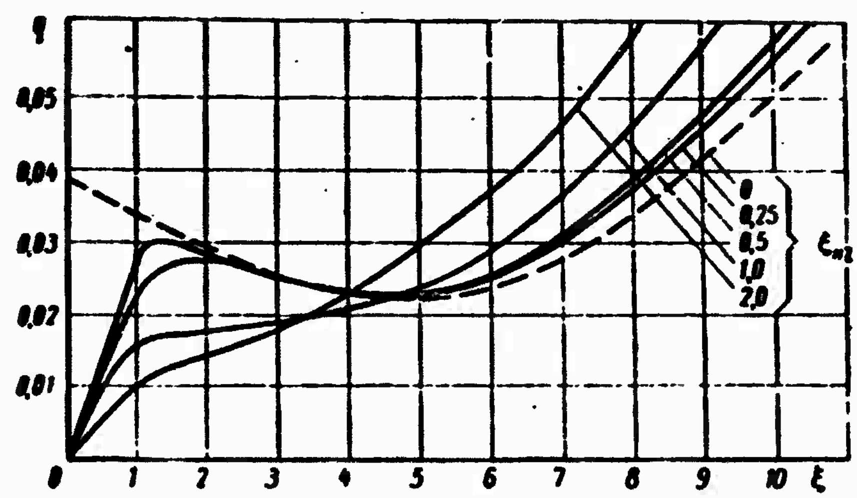



Fig. 48.

In Fig. 48 we depict envelopes $q = f(\xi, \xi_{in})$, plotted various values n and satisfying minimum load values. Curves are calculated for the shell with parameters R/h = 112.5, L/R = 2.45 and various values of six symmetrically located dents $\xi_{in} = 0.25$, 0.5, 1.0, and 2.0. A curve, corresponding to $\xi_{in} = 0$ is built for comparison in the same place. As we can see from the graphic the upper critical load

decreases even with a comparatively small value of the initial bending $\xi_{in}$ = 0.25 or 0.5. Judging from the graph, when $\xi_{in}$ = 1.0 the load increases monotonously, so that the process of deformation of shell should not be accompanied by a pop.

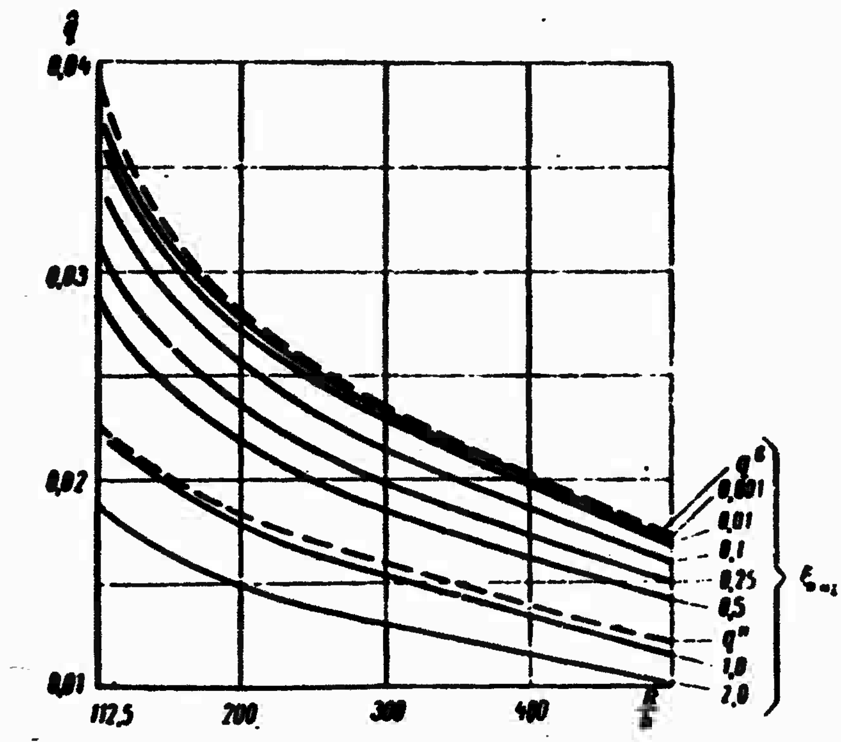



Fig. 49.

In Fig. 49 we show the results of calculations of shells with parameters R/h = 112.5, 200, 500; L/R = 2.45 and the depth of initial bend $\xi_{in}$ = 0.001, 0.01, 0.1, 0.25, 0.5, 1.0 and 2.0.

Judging by the shape and location of curves, we can note that even an insignificant depth of dent $\xi_{in}$ = 0.001 lowers the critical load by 5-8% as compared with the upper critical load. With the increase of the depth of bend the load decreases even more and when $\xi_{in}$ = 0.5 it attains 65-70% of the upper critical value of load corresponding to the smooth shell. Dotted lines on the graphic mark the curves, satisfying the upper and lower critical values of loads for the smooth shell. It is interesting to note that all

curves, satisfying different initial dent values, with which a pop is observed, are located in the region between the upper and lower values of critical loads. Curves, satisfying the depth of the initial dent with which the pop is absent, are located in the region lower than the curve, corresponding to the lower critical value of load.

## § 5. Dynamic Stability of the Cylindrical Shell

Let us assume that a thin-walled circular cylinder with radius r, constant thickness h, and length $l$ is supported by hinges on edges so that one of the hinges has freedom of displacement along the axis, and let us assume that to the end which has the freedom of displacement in moment of time t = 0, load T = const is suddently applied, which is then maintained constant. The sudden application of longitudinal load will produce radial oscillations of the cylinder, where, if load T is less than a certain definite value, these oscillations will occur with non-increasing amplitude near the position of equilibrium, and conversely, if load T is larger than this value, then the amplitude of sag will increase in time and, consequently, the cylindrical shell loses stability.

The problem consists of determining the load (critical), beginning with which the unlimited growth of the amplitude of sag occurs.

Let us consider the axisymmetric loss of stability.* The equation of motion of the element of cylindrical shell has the form,

$$\frac{\partial T_x}{\partial x} = h\rho \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial N_x}{\partial x} + \frac{T_0}{r} - T \frac{\partial^2 w}{\partial x^2} = h\rho \frac{\partial^2 w}{\partial t^2}, \qquad (5.1)$$

$$\frac{\partial M_x}{\partial x} - N_x = 0.$$

*The solution of the problem is given by A. I. Blokhina [67].

Here T = const is the external load, per unit of length; $T_x$, $T_\theta$ are internal axial and circumferential stretching forces respectively, referred to one unit of length; $N_x$ is the axial intersecting force, per one unit of length.

Using known dependencies [15] and introducing them in (5.1), we obtain the equation of motion in displacements,

$$\frac{Eh}{1-\nu^2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}-\frac{\nu}{r}w\right)=h\rho\frac{\partial^2 u}{\partial t^2},$$
$$-\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial x^4}+\frac{Eh}{1-\nu^2}\left(-\frac{w}{r}+\nu\frac{\partial u}{\partial x}\right)-$$
$$-T\frac{\partial^2 w}{\partial x^2}=h\rho\frac{\partial^2 w}{\partial t^2}, \tag{5.2}$$

where u, w are axial and radial displacements.

We assume that longitudinal (axial) displacement is small as compared with transverse ($u \ll w$) displacement and, consequently, the longitudinal inertia can be disregarded. Since $h\rho\frac{d^2u}{dt^2} \to 0$, therefore, from (5.2) we obtain the equation of motion with respect to sag in the radial direction,

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial x^4}+\frac{Eh}{r^2}w+T\frac{\partial^2 w}{\partial x^2}=-h\rho\frac{\partial^2 w}{\partial t^2}. \tag{5.3}$$

We search the solution of equation (5.3), satisfying boundary conditions, in the form

$$w=q(t)\sin\frac{m\pi x}{l}. \tag{5.4}$$

Here q(t) is the amplitude of sag, m is the quantity of half-waves along the generator.

Putting (5.4) in (5.3), for function q(t) we have an ordinary differential equation of the second order

$$\ddot{q}(t)-\alpha q(t)=0, \tag{5.5}$$

where

$$\alpha=\frac{1}{h\rho}\left[T\frac{m^2\pi^2}{l^2}-\frac{Eh^3}{12(1-\nu^2)}\frac{m^4\pi^4}{l^4}-\frac{Eh}{r^2}\right]. \tag{5.5'}$$

The solution of equation (5.5) will be periodic, if $\alpha < 0$, or aperiodic, if $\alpha > 0$. In the latter case the amplitude of sag will increase in time and, consequently, a loss of the shell stability will occur. From condition $\alpha = 0$ we determine the value of critical dynamic force $T^{d}_{cr}$,

$$T^{\text{д}}_{\text{кр}} = \frac{Eh^3}{12(1-\nu^2)} \frac{m^2\pi^2}{l^2} + \frac{Eh}{r^2} \frac{l^2}{m^2\pi^2}. \tag{5.6}$$

[д = d = dynamic] [кр = cr = critical]

It was established that upon a longitudinal impact with force $T$ = const the cylindrical shell loses stability, when the applied force attains the value $T_{cr} = T^{st}_{cr}$, where from a great number of possible axisymmetric forms of loss of stability the shell will be distorted with the formation of half-waves in number $m^*$, i.e., the nearest integer k

$$m = \frac{l}{\pi} \sqrt[4]{\frac{12(1-\nu^2)}{h^2 r^2}}, \tag{5.7}$$

to which corresponds the least value $T^{d}_{cr}$. This form of loss of stability is preferential.

If any external causes will impel the shell to be distorted not in this preferential form, then the critical value of the dynamic load $T^{d}_{cr}$, will be larger than the critical static $T^{st}_{cr}$, where the possibility of regulating the number of half-waves corresponds to the possibility of increase of the dynamic load, which can be withstood by the cylindrical shell without a loss of stability. One of causes, forcing the shell to be distorted with the prescribed number of half-waves, can be the imparting to the shell of initial small amplitude distortions.

It is not difficult to obtain the expression of critical force for the case, when the force $T = T_1 t$, proportional to time, acts on

the shell, and this expression will accurately coincide with (5.6).

Indeed, for function q(t) equation of the type of Bessel's equation takes place,

$$\ddot{q}(z) - \frac{A^3}{B^2} q(z) z = 0. \tag{5.8}$$

Here

$$A = \frac{1}{h\rho}\left[\frac{Eh^3}{12(1-\nu^2)}\frac{m^4\pi^4}{l^4} + \frac{Eh}{r^2}\right],$$
$$B = \frac{1}{h\rho} T_1 \frac{m^2\pi^2}{l^2}, \tag{5.9}$$
$$z = 1 - \frac{B}{A}t.$$

The solution of equation (5.8) will be written in the form

$$q(z) = C_1 z^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B} z^{\frac{3}{2}}\right) + C_2 z^{\frac{1}{2}} I_{-\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B} z^{\frac{3}{2}}\right). \tag{5.10}$$

when z > 0, i.e., in the initial time interval and

$$q(\tilde{z}) = \tilde{C}_1 \tilde{z}^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B} \tilde{z}^{\frac{3}{2}}\right) + \tilde{C}_2 \tilde{z}^{\frac{1}{2}} I_{-\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B} \tilde{z}^{\frac{3}{2}}\right), \tag{5.11}$$

where $\tilde{z} = \frac{B}{A}t - 1$

with the time increase. From properties of Bessel functions it ensues that the shell in the initial moment of time when z > 0 oscillates near the position of equilibrium, and when z < 0 loses stability, i.e., the sag increases in time. From the condition z = 0 the critical moment of time $T_{cr}$, is determined and consequently, the critical force, i.e.,

$$T^{д}_{кр} = T_1 t_{кр} = \frac{Eh^3}{12(1-\nu^2)}\frac{m^2\pi^2}{l^2} + \frac{Eh}{r^2}\frac{l^2}{m^2\pi^2},$$

similarly to the case, when T = const.

Lastly, we consider the case when the cylindrical shell has the initial distortion

$$w_0 = k \sin\frac{m\pi x}{l}, \tag{5.12}$$

where k is the initial amplitude, m is the initial quantity of half-waves. Let us assume that on the end of the shell constant force T = const is applied momentarily. The motion equation (5.3) in this case assumes the form

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4(w-w_0)}{\partial x^4}+\frac{Eh}{r^2}(w-w_0)+T\frac{\partial^2 w}{\partial x^2}=-h\rho\frac{\partial^2 w}{\partial t^2}. \qquad (5.13)$$

Let us assume that the additional sag occurs in the same form as the initial, i.e.,

$$w_1=q(t)\sin\frac{m\pi x}{l}. \qquad (5.14)$$

Then, according to (5.12) and (5.14),

$$w=w_0+w_1=[k+q(t)]\sin\frac{m\pi x}{l}. \qquad (5.15)$$

For function q(t) after introduction of (5.15) in (5.13) and simple transformations we obtain,

$$\ddot{q}(t)-\alpha q(t)=\frac{k}{h\rho}T\frac{m^2\pi^2}{l^2}. \qquad (5.16)$$

Here $\alpha$ has the same expression as (5.5').

Reasonings, similar to those used in the beginning, lead to the conclusion, that the initial sag does not have any effect on the value of critical dynamic force, if the initial quantity of half-waves m will coincide exactly with value $m^*$, with which value $T^{st}_{cr}$ is obtained If, however, $m \neq m^*$, then value $T^{d}_{cr} > T^{st}_{cr}$, since here we apply, in a way, additional bonds, and the shell will be stabler.

Thus, by prescribing a small bend we can make the shell stabler.

We also note that if $T = T_1 t$, then, reasoning, in a manner analogous to an earlier reasoning, we will also attain the equation of the type of Bessel's equation,

$$\ddot{q}(z)+\frac{A^2}{B^2}q(z)z=\frac{A^2}{B^2}k, \qquad (5.17)$$

where z has the same value as before. The solution for $z > 0$ has the form

$$q(z)=c_1 z^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}z^{\frac{3}{2}}\right)+c_2 z^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}z^{\frac{3}{2}}\right)+$$

$$+\frac{A^2}{B^2}\frac{k}{w}\left[z^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}z^{\frac{3}{2}}\right)\int_0^z S^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right)dS-\right.$$

$$\left.-z^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}z^{\frac{3}{2}}\right)\int_0^z S^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right)dS\right], \qquad (5.18)$$

where w is Wronskian determinant; $w = f_1 f_2' - f_2 f_1'$, $f_1$ and $f_2$ are particular integrals of equation (5.17).

Constants $c_1$ and $c_2$ are determined from initial conditions, when $t = 0$, $q(0) = k$, $q(0) = 0$.

$$c_1=-\frac{1}{w}\left(\frac{A^{\frac{3}{2}}}{B}R_1 I_{\frac{2}{3}}+R_2 I_{-\frac{1}{3}}\right),$$

$$c_2=-\frac{1}{w}\left(-R_2 I_{\frac{1}{3}}+\frac{A^{\frac{3}{2}}}{B}R_1 I_{-\frac{2}{3}}\right). \qquad (5.19)$$

Here

$$R_1=-\frac{k}{w}\frac{A^3}{B^3}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\right)\int_0^1 S^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right)dS-\right.$$

$$\left.-I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\right)\int_0^1 S^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right)dS\right]+k,$$

$$R_2=\frac{k}{w}\frac{A^{\frac{9}{2}}}{B^3}\left[I_{\frac{2}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\right)\int_0^1 S^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right)dS+\right.$$

$$\left.+I_{-\frac{2}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\right)\int_0^1 S^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right)dS\right]. \qquad (5.20)$$

With the increase of time t argument z becomes negative. Introducing variable $\tilde{z}=\frac{B}{A}t-1$, we obtain the equation of the type of Bessel's equation,

$$\ddot{q}(\tilde{z}) - \frac{A^3}{B^3} q(\tilde{z})\tilde{z} = \frac{A^3}{B^3} k. \tag{5.21}$$

The solution of this equation will be written through modified Bessel functions,

$$\begin{aligned} q(\tilde{z}) = {} & \tilde{c}_1 \tilde{z}^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\tilde{z}^{\frac{3}{2}}\right) + \\ & + \tilde{c}_2 \tilde{z}^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\tilde{z}^{\frac{3}{2}}\right) + \\ & + \frac{A^3}{B^3}\frac{k}{w}\Big[\tilde{z}^{\frac{1}{2}} I_{-\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}\tilde{z}^{\frac{3}{2}}\right)\int_0^{\tilde{z}} S^{\frac{1}{2}} \times \\ & \times I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right) dS - \tilde{z}^{\frac{1}{2}} I_{\frac{1}{3}}\left(\frac{2}{3}\frac{A^3}{B}\tilde{z}^{\frac{3}{2}}\right)\int_0^{\tilde{z}} S^{\frac{1}{2}} \times \\ & \times I_{-\frac{1}{3}}\left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{B}S^{\frac{3}{2}}\right) dS\Big]. \end{aligned} \tag{5.22}$$

Constants $\tilde{c}_1$ and $\tilde{c}_2$ are determined from the condition, that when $t = 0$ and $\tilde{z} = 0$ particle velocities and sags have to coincide. Then we obtain that

$$c_1 = -\tilde{c}_1, \quad c_2 = \tilde{c}_2. \tag{5.23}$$

Thus, the amplitude of sag is determined fully. That moment of time, when the amplitude of sag of the shell passes from oscillatory motions to growth, we will term the critical moment, and the load, corresponding to it, — the critical load

$$T_{\text{кр}} = T_1 t_{\text{кр}}. \tag{5.24}$$

We should bear in mind that the obtained solutions are true only for small sags $\left(\frac{h}{r} \geq w\right)$. An analysis of the solution allows us to conclude that:

1) the greater the speed of the load, the less the amplitude

of the sag,

2) the greater the speed of load and the smaller the initial sag, the higher the overload factor $T^{d}_{cr}/T^{st}_{cr}$;

3) the minimum load, endured by the shell, is obtained for the same quantity of half-waves for any speed of the load.

We can indicate another method of setting up equations of motion of thin shells and solution of the problem of the dynamic stability of cylinder [47], proceeding from the Hamilton-Ostrogradskiy principle,

$$\delta I=\int_{t_0}^{t_1}\delta L\,dt=0,\quad \delta L=\delta T+\delta A+\delta E-\delta W. \tag{5.25}$$

From now on, $\delta T$ is the variation of kinetic energy of shells, $\delta A$ is the variation of work of external forces and moments, acting on the shell, $\delta W$ is the variation of shell deformation work, $\delta E$ is the variation of work of force factors, which depend on speed and produce damping of the motion $u_1$, $u_2$, $w$ are projections on directions of unit vectors; $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$ are vectors of load displacement which brings the middle surface $\sigma^0$ to surface $\sigma'$; $e_i^*$, $m_i^*$, ... are values $e_i$, $m_i$, ... in the new deformed state; $e_{ix}$, $\omega_i$, $E_i$, $E_3$ are angles of rotation of coordinate vectors $\bar{r}$, $l$, $\bar{m}$ in the process of deformation, $i=\frac{d()}{d\alpha_i}$ are partial derivatives. Vectors of speed and acceleration of points of the middle surface in the case of small deformations have the form,

$$\left\{\begin{aligned}\frac{\partial^2\bar{v}}{\partial t^2}&=\sum_{i,j=1}^{2}\Big[\bar{e}_i^*\Big(\frac{\partial^2u_i}{\partial t^2}+e_{ij}\frac{\partial^2u_i}{\partial t^2}+w_i\frac{\partial^2w}{\partial t^2}\Big)-\\&\quad-\bar{m}^*w_i\frac{\partial^2u_i}{\partial t^2}\Big]+m^*\frac{\partial^2w}{\partial t^2}\qquad i\neq j,\\\frac{\partial\bar{v}}{\partial t}&=\sum_{i,j=1}^{2}\Big[\bar{e}_i^*\Big(\frac{\partial u_i}{\partial t}+e_{ij}\frac{\partial u_j}{\partial t}+w_i\frac{\partial w}{\partial t}\Big)-\\&\quad-\bar{m}^*w_i\frac{\partial u_i}{\partial t}\Big]+\bar{m}^*\frac{\partial w}{\partial t}.\end{aligned}\right. \tag{5.26}$$

Converting δA and δW, we obtain,

$$\int\limits_{t_0}^{t_1}\iint \Big\{\sum_{i=1}^{2}\big[(F_i + Y_i)\,\delta u_i - (M_i + Z_i)\,\delta w_i\big] + (F_3 + Y_3)\,\delta w\Big\} \times$$
$$\times\, d\alpha_1\, d\alpha_2\, dt + \int\limits_{t_0}^{t_1}\int \Big\{\sum_{i=1}^{2} R_i\,\delta u_i + R_3\,\delta w -$$
$$- (\overline{G} - G)\,\delta w_n\Big\}\, dS\, dt = 0. \tag{5.27}$$

where

$$F_1 = (A_2 \overset{\circ}{T}_{11})_{,1} + (A \overset{\circ}{T}_{21})_{,2} + \overset{\circ}{T}_{12} A_{1,2} - \overset{\circ}{T}_{22} A_{2,1} +$$
$$+ A_1 A_2 (\overset{\circ}{N}_2 \overset{\circ}{K}_{11} + \overset{\circ}{N}_2 \overset{\circ}{K}_{12} + \overset{\circ}{X}_1) \qquad \overleftarrow{1,2}$$
$$Y_1 = A_1 A_2 \varepsilon h \left(\frac{\partial u_1}{\partial t} + e_{12}\frac{\partial u_2}{\partial t} + w_1 \frac{\partial w}{\partial t}\right) -$$
$$- A_1 A_2 \rho h \left(\frac{\partial^2 u_1}{\partial t^2} + e_{12}\frac{\partial^2 u_2}{\partial t^2} + w_1 \frac{\partial^2 w}{\partial t^2}\right),$$
$$F_3 = (A_2 \overset{\circ}{N}_1)_{,1} + (A_1 \overset{\circ}{N}_2)_{,2} -$$
$$- A_1 A_2 (\overset{\circ}{T}_{11} \overset{\circ}{K}_{11} + \overset{\circ}{T}_{22} \overset{\circ}{K}_{22} + \overset{\circ}{K}_{12} \overset{\circ}{T}_{12} + \overset{\circ}{K}_{12} \overset{\circ}{T}_{21} - X_3), \tag{5.28}$$
$$Y_3 = A_1 A_2 \varepsilon h \left(\frac{\partial w}{\partial t} - w_1 \frac{\partial u_1}{\partial t} - w_2 \frac{\partial u_2}{\partial t}\right) -$$
$$- A_1 A_2 \rho h \left(\frac{\partial^2 w}{\partial t^2} + w_1 \frac{\partial^2 u_1}{\partial t^2} - w_2 \frac{\partial^2 u_2}{\partial t^2}\right),$$
$$M_i = (A_2 \overset{\circ}{M}_{11})_{,1} + (A_1 \overset{\circ}{M}_{21})_{,2} + \overset{\circ}{M}_{22} A_{1,2} - \overset{\circ}{M}_{22} A_{2,1} +$$
$$+ A_1 A_2 (L^{\circ} - \overset{\circ}{N}_1),$$
$$Z_i = A_1 A_2 \frac{h^3}{12}\left(\rho \frac{\partial^2 w_i}{\partial t^2} - \varepsilon \frac{\partial w_i}{\partial t}\right),$$
$$R_i = \Phi_s^i + \Phi_s^3 u_i - \sum_{k=1}^{2} \overset{\circ}{T}_{ik} n_k, \quad R_3 = \Phi_3^3 - N + \frac{dH}{ds} - \sum_{i,k}^{2} T_{ik} w_i n_k.$$

ε is the experimental coefficient.

The relationship (5.27) is the Bubnov-Galerkin method equation, which enables us to integrate approximately dynamic equations of shell motion. Hence, the shell motion equation,

$$F_i + Y_i = 0; \quad F_3 + Y_3 = 0;$$
$$M_i + Z_i = 0 \tag{5.29}$$

and static boundary conditions, $R_i = 0$, $R_3 = 0$, $\overline{g} = g$.

Let us consider particular cases.

1) Linear oscillations of shells with a slight bend. In this case displacements are small, squares of displacements and their derivatives with respect to coordinates can be disregarded, so that dynamic equations and static boundary conditions are simplified. In expressions $F_i$ and $F_3$ we replace curvatures of coordinates lines $k^*_{ij}$ by their initial values $k_{ij}$, and inertial terms and expression of unbalanced contour forces are linearized,

$$Y_i = A_1A_2\left(\varepsilon\frac{\partial u_i}{\partial t} - \rho\frac{\partial^2 u_i}{\partial t^2}\right)h_1,\ Y_3 = A_1A_2h\left(\varepsilon\frac{\partial w}{\partial t} - \rho\frac{\partial^2 w}{\partial t^2}\right), \tag{5.30}$$

$$R_i = \Phi^i_s - \sum_{k=1}^{2} T^0_{ik}n_k,\quad R_3 = \Phi^3_s - N + \frac{\partial H}{\partial s}. \tag{5.31}$$

Actually, considering (5.28), equation $F_3 + Y_3 = 0$ can be written in the form

$$F_3 + A_1A_2\varepsilon h\left(\frac{\partial w}{\partial t}\right) - A_1A_2\rho h\frac{\partial^2 w}{\partial t^2} + \sum_{i=1}^{2} w_iY_i = 0,$$

where terms, containing $w_i$, constitute the effect of longitudinal forces of inertia. For $F_3$ and $F_i$ these estimates take place

$$F_3 \sim Eh^3\frac{r^4}{R^4},$$

$$F_i \sim Eh\frac{r}{R^2},$$

where r is max $\{n, \lambda\}$; n is the number of transverse waves, $\lambda = \pi R\frac{m}{l}$; m is the number of longitudinal half-waves. If $w \sim h^2\frac{r}{R^2}$, $r^2 \gg 1$, then, as we see from the preceding estimate, it is possible to disregard $\Sigma w_iF_i$ as compared to $F_3$.

2) Nonlinear oscillations with average bend. The bend is called an average bend, if $u_i \sim e_{ik} \sim \varepsilon_p$, but terms of bend origin $w \sim w_i \sim x_{ik} \sim \sqrt{\varepsilon p}$, $w_i^2 \ll 1$ during all the time of motion. $F_i$, $F_3$ and curvatures $k^*_{ij}$ are replaced by their expressions according to the

linear theory,

$$
\begin{aligned}
k^{*}_{11} &= k_{11}(1+e_{22}) - k_{12}e_{21} - \frac{1}{A_1}\frac{\partial w_1}{\partial a_1} - \frac{w_2}{A_1A_2}\frac{\partial A_1}{\partial a_2},\\
k^{*}_{12} &= k_{12}(1-e_{11}) - k_{11}e_{12} - \frac{1}{A}\frac{\partial w_2}{\partial a_1} + \frac{w_1}{A_1A_2}\frac{\partial A_1}{\partial a_2},
\end{aligned}
\tag{5.32}
$$

$$\overleftrightarrow{1\,2}$$

and are expressed by formulas (5.28), and in expression $Y_i$ terms, containing $e_{12}$ and $e_{21}$ are discarded, while $Y_3$ remains without change.

3) Nonlinear oscillations of sloping shells with the average bend. Here following simplifications are possible; firstly we disregard the effect of tangential displacements $u_i$ on angles of rotation $w_i$, i.e., we consider that $w_i = \frac{1}{A_i}\cdot\frac{\partial w}{\partial a_i}$; secondly, in expressions $F_1$ and $F_2$ we reject terms, containing forces $N_1^*$ and $N_2^*$; thirdly we disregard the effect of angles of turn $e_{ik}$ on bend deformations, i.e., in expressions of curvatures (5.32) we discard terms, containing $e_{ik}$.

The set up dynamic equations of nonlinear theory of shells are applicable to short shells and shells of average length.

Let us turn to setting up of differential equations of the dynamic stability of shells.

Let us consider two consecutive states of the shell. Let us assume that the middle surface σ with the help of displacement $\bar{v}^{\prime}=u_1^{\prime}\bar{e}_1+u_2^{\prime}\bar{e}_2+w^{\prime}m$ passes into surface $\sigma_1$. Knowning $\bar{v}^{\prime}$, we can determine the values, characterizing deformation and stresses, which satisfy motion equations,

$$F_i^{\prime}+Y_i^{\prime}=0,\quad M_i^{\prime}+Z_i^{\prime}=0,\quad F_3^{\prime}+Y_3^{\prime}=0. \tag{5.33}$$

With a certain value of time or certain relationships of motion [14] parameters another state is possible along with the motion, designated with index "1". Motion equations, corresponding to this

state, are termed equations of indifferent motion, and parameters — critical parameters. Motion parameters imply the external load, time, oscillation frequencies, etc.

Let us assume that the vector of additional displacement $\overline{v} = u_1\overline{e}_1 + \overline{u}_2 e_2 + w\overline{m}$ changes surface $\sigma'$ into $\sigma^*$, where $\overline{v}^* = \overline{v}' + \overline{v}$. Let us determine the values, characterizing deformation and stress during displacment of $\overline{v}^*$. Let us assume that value $\overline{v}$ is the first order of smallness, and $\overline{v}'$ is a finite value (for instance, of the order of a unity). In values, characterizing deformation and stress, we refain terms of the first order of smallness together with the finite terms:

$$T_{11}^{\circ} = k(\varepsilon_{11}^{\circ} + \gamma\varepsilon_{22}^{\circ}) = T_{11}' + T_{11}; \quad T_{11}' = k(\varepsilon_{11}' + \gamma\varepsilon_{22}');$$
$$T_{11} = k(\varepsilon_{11} + \gamma\varepsilon_{22}); \ \varepsilon_{11}^{\circ} = e_{11}^{\circ} + \frac{1}{2}(e_{11}^{\circ 2} + e_{12}^{\circ 2} + w_1^{\circ 2}) = \varepsilon_{11}' + \varepsilon_{11}, \tag{5.34}$$

where

$$\varepsilon_{11}' = e_{11}' + \frac{1}{2}(e_{11}'^2 + e_{12}'^2 + w_1'^2),$$
$$\varepsilon_{11} = (1 + e_{11}')e_{11} + e_{12}'e_{12} + w_1'w_1.$$

Equations of motion in indifferent state,

$$F_i^{\circ} + Y_i^{\circ} = 0, \quad F_3^{\circ} + Y_3^{\circ} = 0, \quad Z_i^{\circ} + M_i^{\circ} = 0. \quad i = 1,2. \tag{5.35}$$

Now, substracting from equation (5.35) equation (5.33), we obtain the equation of dynamic stability,

$$[A_2e_{22}T_{11}' + A_2(1 + e_{22}')T_{11}]_{,1} + (A_1T_{21}')_{,2} + (T_{12}'A_{1,2}) - T_{22}'(A_2e_{22})_{,1} -$$
$$- T_{22}[A_2(1 + \varepsilon_{22}')]_{,1} + A_1A_2[N_1'x_{11} + N_1(k_{11} + x_{11}') + N_2'x_{12} +$$
$$+ N_2(k_{12} + x_{12}') + x_1^{\circ} - x_1' + \varepsilon h\frac{\partial u_1}{\partial t} + e_{12}\frac{\partial u^*}{\partial t^2} + e_{12}'\frac{\partial u_2}{\partial t} +$$
$$+ w_1\frac{\partial w'}{\partial t} + w_1'\frac{\partial w}{\partial t}\Big)\Big] = A_1A_2\varrho h\left\{\frac{\partial^2 u_1}{\partial t^2} + e_{12}\frac{\partial^2 u_2}{\partial t^2} + e_{12}'\frac{\partial^2 u_2}{\partial t^2} +\right.$$
$$\left. + w_1\frac{\partial^2 w_1'}{\partial t^2} + w_1'\frac{\partial^2 w}{\partial t^2}\right\}. \quad \underset{\leftarrow}{\overset{\rightarrow}{1,2}} \tag{5.36}$$

$$(A_2N_1')_{,1} + (A_1N_2')_{,2} - A_1A_2[T_{11}'\varkappa_{11}' + T_{11}'(k_{11} + \varkappa_{11}') + 2T_{12}'\varkappa_{12}' +$$
$$+ 2T_{12}'(k_{12} + \varkappa_{12}') + T_{22}'\varkappa_{22}' + T_{22}'(k_{22} + \varkappa_{22}') - \dot{\varkappa}_3' - \varkappa_3' -$$
$$- \varepsilon h\left(- w_1'\frac{\partial u_1}{\partial t} - w_1\frac{\partial u_1'}{\partial t} - w_2'\frac{\partial u_2}{\partial t} - w_2'\frac{\partial u_2'}{\partial t} + \frac{\partial w}{\partial t}\right)\Big] =$$
$$= \rho hA_1A_2\left\{- w_1'\frac{\partial^2 u_1}{\partial t^2} - w_1\frac{\partial^2 u_1'}{\partial t^2} - w_2'\frac{\partial^2 u_2}{\partial t^2} - w_2'\frac{\partial^2 u_2'}{\partial t^2} + \frac{\partial^2 w}{\partial t^2}\right\}; \qquad (5.37)$$
$$M_t + Z_t = 0. \qquad (5.38)$$

The static part corresponds to equations of indifferent state [47].

Let us consider the dynamic stability of a cylindrical shell of average length, when $l \sim \pi R$. To such shells, we can apply the theory of sloping shells. Let us assume that the initial position is zero-moment or almost zero-moment. Then $\frac{1}{2}\,hx_{ij}' \ll w^1$, $aw_i'$, wi will be values of one order with elongations, i.e., we shall assume that speeds and accelerations of points of the shell are small and can be disregarded.

We shall limit ourselves to considering linear oscillations of the cylindrical shell. The deformation components are expressed by formulas,

$$\varkappa_{ik} = -w_{,ik}; \quad \varepsilon_1 = u_{,1}; \quad \varepsilon_2 = v_{,2} + \frac{w}{R}; \quad 2\varepsilon_{12} = u_{,2} + v_{,1}. \qquad (5.39)$$

However, in the initial zero-moment state forces will be expressed by formulas,

$$T_1' = -N; \quad T_{12}' = T; \quad T_2' = -pR, \qquad (5.40)$$

where N is the intensity of axial compression, p is external pressure, and T is the shear force.

Equations of dynamic stability in displacement have form,

$$u_{,11} + \frac{1-\gamma}{2}u_{,22} + \frac{1+\gamma}{2}v_{,12} + \frac{\gamma}{R}w_{,1} + x_1 = \rho h/k\ddot{u} - \frac{\varepsilon h}{k}\dot{u},$$
$$u_{,22} + \frac{1-\gamma}{2}v_{,11} + \frac{1+\gamma}{2}u_{,12} + 1/Rw_{,2} + x_2 = \rho h/k\ddot{v} - \frac{\varepsilon h}{k}\dot{v}, \qquad (5.41)$$

$$D\Delta\Delta w-\rho\frac{h^3}{12}\Delta\ddot{w}+\Phi+k/R(v_{,2}+\gamma u_{,1}+1/Rw)+$$

$$+\rho h\ddot{w}-\varepsilon h\dot{w}-x_3=0, \qquad (5.41\ \text{con't})$$

where

$$\Phi=Nw_{,11}+2T_{,12}+pRw_{,22};\quad \Delta(\ )=(\ )_{,11}+(\ )_{,22}.$$

$x_i$ are certain perturbations with respect to axes, and $\rho\frac{h^3}{12}\Delta\ddot{W}$ is the term which appeared in veiw of calculation of the rotation inertia force $w_{,1}=\frac{\partial w}{\partial x}$, $\dot{w}=\frac{\partial w}{\partial t}$. First, let us consider the problem of axi-symmetric oscillation, when $v = 0$, $u = u(x, t)$, $w = w(x, t)$. Expressing the motion equation through one displacement (see general case of oscillation) for different assumptions,

$$1)\ \ddot{u}\neq 0,\quad \ddot{w}\neq 0,\quad 2)\ \ddot{u}=0,\quad \ddot{w}\neq 0,\quad 3)\ \ddot{u}\neq 0,\quad \ddot{w}=0$$

we obtain respectively,

$$\frac{\partial^4}{\partial\tau^4}\left\{w-h^2/12R^2\frac{\partial^2 w}{\partial\alpha^2}\right\}+\frac{\partial^2}{\partial\tau^2}\left\{2\frac{h^2}{12R^2}\cdot\frac{\partial^4 w}{\partial\alpha^4}+\frac{N}{k}\frac{\partial^2 w}{\partial\alpha^2}-\right.$$

$$\left.-\frac{\partial^2 w}{\partial\alpha^2}-w\right\}-\left\{\frac{h^2}{12R^2}\frac{\partial^6 w}{\partial\alpha^6}+\frac{\partial^2}{\partial\alpha^2}\left(\frac{N}{k}\cdot\frac{\partial^2 w}{\partial\alpha^2}\right)+\right.$$

$$\left.+(1-\gamma)^2\frac{\partial^2 w}{\partial\alpha^2}\right\}=0; \qquad (5.42)$$

$$\frac{\partial^2}{\partial\tau^2}\left\{\frac{\partial^2 w}{\partial\alpha^2}-\frac{h^2}{12R^2}\frac{\partial^4 w}{\partial\alpha^4}\right\}+\frac{h^2}{12R^2}\frac{\partial^6 w}{\partial\alpha^6}+\frac{\partial^2}{\partial\alpha^2}\left(\frac{N}{k}\frac{\partial^2 w}{\partial\alpha^2}\right)+$$

$$+(1-\gamma^2)\frac{\partial^2 w}{\partial\alpha^2}=0. \qquad (5.43)$$

$$\frac{h^2}{12R^2}\cdot\frac{\partial^6 u}{\partial\alpha^6}+\frac{1}{k}\frac{\partial^2}{\partial\alpha^2}\left(N\frac{\partial^2 u}{\partial\alpha^2}\right)+(1-\gamma^2)\frac{\partial^2 u}{\partial\alpha^2}=$$

$$=\frac{\partial^2}{\partial\tau^2}\left\{\frac{h^2}{12R^2}\cdot\frac{\partial^4 u}{\partial\alpha^4}+\frac{N}{k}\frac{\partial^2 u}{\partial\alpha^2}+u\right\}; \qquad (5.44)$$

when

$$\varepsilon=0\ \text{and}\ x_i=0;\quad \tau=t\left(\frac{k}{\rho h}\right)^{\frac{1}{2}}\frac{1}{R},\quad x=\alpha R.$$

In each of these cases we define, as an example, natural oscillation frequencies for the hinged fastening of edges, selecting displacements in the form,

$$w=\sum A_m(\tau)\sin\lambda\alpha;\quad u=\sum B_m(\tau)\cos\lambda\alpha,\ \text{where}\ \lambda=m\frac{\pi R}{l}.$$

$A_m$ and $B_m$ we take in the form,

$$A_m(\tau) = A_0 \begin{cases} \sin \omega\tau \\ \cos \omega\tau \end{cases}; \qquad B_m(\tau) = B_0 \begin{cases} \sin \omega\tau \\ \cos \omega\tau \end{cases},$$

where $A_0$ and $B_0$ are constants, and $\omega$ is a dimensionless angular frequency.

Then from (5.42)-(5.44) we obtain for N = const,

$$\omega^2_{(3,4)1,2} = \frac{\lambda^2+1}{2} \pm \left\{\left(\frac{\lambda^2+\lambda}{2}\right)^2 - \lambda^2\left[\frac{h^2}{12R^2}\lambda^4 - \frac{N}{k}\lambda^2 + (1-\gamma^2)\right]\right\}^{1/2}; \tag{5.45}$$

$$\omega^2_{(3,5)} = \left\{\frac{h^2}{12R^2}\lambda^4 - \frac{N}{k}\lambda^2 + (1-\gamma^2)\right\}; \tag{5.46}$$

$$\omega^2_{(3,6)} = \lambda \left\{1 - \gamma^2\left(\frac{h^2}{12R^2}\lambda^4 - \frac{N}{k}\lambda^2 + 1\right)^{-1}\right\},$$

where $\omega_{(3,4)1}$ is the natural frequency of longitudinal oscillations taking into account transverse inertia forces; $\omega_{(3,4)2}$ is the natural frequency of transverse oscillations taking into account longitudinal inertia forces, $\omega_{(3,5)}$ is the natural frequency of transverse oscillations without calculation of longitudinal inertia forces; $\omega_{(3,6)}$ is the natural frequency of longitudinal oscillations without calculation of transverse forces of inertia. Above we assumed that the inertial force of rotation can be disregarded, if $m \lesssim \left(\frac{R}{hR}\right)^{1/2}$. It is known that $\frac{N}{K} \sim \frac{h}{R}$. This enables us in (5.42) to disregard $\frac{N}{K}\lambda^2$ as compared to $\lambda^2$. In expressions (5.45), (5.46) we leave $\frac{N}{K}$, since for $\lambda \sim \sqrt{R/h}$ the preliminary compression and extension can have an essential influence on the frequency of natural oscillations.

Expression $\omega^2_{(3,5)}$ has an extremum for $\lambda = 0$ and $\lambda^2 = \frac{N}{K}\cdot\frac{6R^2}{h^2}$ respectively;

$$\omega^2_{(3,5)\min} = (1-\gamma^2), \quad \omega^2_{(3,5)2\min} = (1-\gamma^2)\left[1 - \left(\frac{N}{N_{kp}}\right)^2\right]. \tag{5.47}$$

The load changes the least frequency in the direction of decrease. From (5.47) it is clear that the effect of preliminary tension becomes significant when $N \sim N_{cr}$, and when $N \sim \sqrt{\frac{h}{R}} N_{cr}$ its effect may be disregarded.

Longitudinal forces of inertia affect transverse oscillations frequencies in the direction of decrease and become significant with the increase of length. For instance, when $l$ = 120 cm, m = 1 (i.e., for the least frequency) $\omega^2_{(3,5)} \cdot \omega^2_{(3,4)2} = 3.7666$. If it is permissible to determine the square of frequency with the error up to 6%, then formula (5.46) can be applied when m = 1 to $l$ = 2R, when m = 2 to $l$ = 4R etc. In joint calculation of longitudinal and transverse forces of inertia the natural frequency of transverse vibrations may be less $1 - \gamma^2$. Transverse forces of inertia affect longitudinal vibration frequencies in the direction of increase.

Thus, in solving the problems of axisymmetric vibration it is not always possible to disregard the effect of longitudinal forces of inertia on transverse oscillations, and vice versa.

We return to the general equation (5.41). We introduce auxiliary functions,

$$\gamma \frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} = Q(x, s, t); \quad \frac{\partial u}{\partial s} - \frac{\partial v}{\partial x} = f(x, s, t). \tag{5.48}$$

We multiply the first equation of (5.41) by $\gamma$, differentiate with respect to x, then the second equation we differentiate with respect to s and add the results. We obtain,

$$\nabla Q + \frac{1-\gamma}{2} \frac{\partial^2 f}{\partial x \partial s} + \frac{\gamma^2}{R} \frac{\partial^3 w}{\partial x^3} + \frac{1}{R} \frac{\partial^3 w}{\partial s^3} = \\ = \frac{\rho h}{k} \frac{\partial^2}{\partial t^2} Q - \frac{\epsilon h}{k} \frac{\partial}{\partial t} Q. \tag{5.49}$$

Further, differentiating the first equation of (5.41) with respect

to s, and the second equation with respect to t and substracting the second from the first, we find,

$$L(f)=\frac{2}{R}\frac{\partial^2 w}{\partial x\,\partial s}.$$

where

$$L(\ )=\nabla(\ )-\frac{2\rho h}{k(1-\gamma)}\frac{\partial^2}{\partial t^2}(\ )+\frac{2\varepsilon h}{k(1-\gamma)}\frac{\partial}{\partial t}(\ ). \tag{5.50}$$

Applying operator (5.50) to expression (5.49), we obtain,

$$\begin{aligned}L_1(Q)=&-\frac{\gamma^2}{R}\frac{\partial^2}{\partial x^2}\left\{\nabla w-\frac{2\rho h}{k(1-\gamma)}\cdot\frac{\partial^2 w}{\partial t^2}+\frac{2\varepsilon h}{k(1-\gamma)}\frac{\partial w}{\partial t}\right\}-\\&-\frac{1}{R}\frac{\partial^2}{\partial s^2}\left\{\nabla w-\frac{2\rho h}{k(1-\gamma)}\frac{\partial^2 w}{\partial t^2}+\frac{2\varepsilon h}{k(1-\gamma)}\frac{\partial w}{\partial t}\right\}-\\&-\frac{1-\gamma^2}{R}\cdot\frac{\partial^4 w}{\partial x^2\,\partial s^2},\end{aligned} \tag{5.51}$$

where $L_1$ is the new operator,

$$\begin{aligned}L_1(\ )=&\nabla\nabla(\ )-\frac{3-\gamma}{1-\gamma}\frac{\rho h}{k}\frac{\partial^2}{\partial t^2}\nabla(\ )+\frac{3-\gamma}{1-\gamma}\frac{\varepsilon h}{k}\frac{\partial}{\partial t}\nabla(\ )+\\&+\frac{2}{1-\gamma}\left(\frac{\rho h}{k}\right)^2\frac{\partial^4}{\partial t^4}(\ )-\frac{4}{1-\gamma}\frac{\rho\varepsilon h^2}{k^2}\frac{\partial^3}{\partial t^3}(\ )+\\&+\left(\frac{\varepsilon h}{k}\right)^2\frac{2}{1-\gamma}\frac{\gamma^2}{\partial t^2}(\ ).\end{aligned}$$

Applying it to the third equation from (5.4) when $x_1 = x_2 = 0$, we obtain,

$$\begin{aligned}&\left\{\frac{h^2}{12}\nabla\nabla\nabla\nabla w+\frac{1}{k}\nabla\nabla\Phi+(1-\gamma^2)\frac{\partial^4 w}{\partial x^4}-\frac{1}{k}\nabla\nabla x_3\right\}+\\&+\frac{\varepsilon h}{k}\frac{\partial}{\partial t}\left\{-\nabla\nabla w+\frac{h^2}{12}\frac{3-\gamma}{1-\gamma}\nabla\nabla\nabla w+\frac{1}{k}\frac{3-\gamma}{1-\gamma}\nabla\Phi+\right.\\&+(3+2\gamma)\frac{1}{R^2}\frac{\partial^2 w}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2 w}{\partial s^2}-\left.\frac{1}{k}\frac{3-\gamma}{1-\gamma}\nabla x_3\right\}+\\&+\frac{\rho h}{k}\frac{\partial^2}{\partial t^2}\left\{\nabla\nabla w-\frac{2(2-\gamma)}{(1-\gamma)}\frac{h^2}{12}\nabla\nabla\nabla w-\frac{1}{k}\frac{3-\gamma}{1-\gamma}\nabla\Phi-\right.\\&-\frac{1}{R^2}\frac{\partial^2 w}{\partial s^2}-(3+2\gamma)\frac{\partial^2 w}{\partial x^2}+\frac{1}{k}\cdot\frac{3-\gamma}{1-\gamma}\nabla x_3+\\&+D\left(\frac{\varepsilon h}{k}\right)^2\frac{1}{\rho h}\frac{2}{(1-\gamma)}\nabla\nabla w+\left(\frac{\varepsilon h}{k}\right)^2\frac{1}{\rho h}\cdot\frac{2}{1-\gamma}\Phi+\\&+\frac{1}{R^2}\frac{2}{1-\gamma}\frac{\varepsilon^2 h}{kp}w-\frac{3-\gamma}{1-\gamma}\cdot\frac{\varepsilon^2 h}{kp}\nabla w-\left.\left(\frac{\varepsilon h}{k}\right)^2\frac{2}{1-\gamma}\frac{1}{\rho h}x_3\right\}+\\&+\frac{\rho h}{k}\frac{\varepsilon h}{k}\frac{\partial^3}{\partial t^3}\left\{2\frac{3-\gamma}{1-\gamma}\Delta w+\frac{1}{1-\gamma}\frac{x_3}{k}-\frac{7-\gamma}{1-\gamma}\frac{h^2}{12}\nabla\nabla w\times\right.\\&\times\frac{4}{1-\gamma}\cdot\frac{1}{k}\Phi-\frac{2}{R^2(1-\gamma)}\left.\left(2+\frac{k\varepsilon^2R^2}{\rho h}\right)w\right\}+\end{aligned} \tag{5.52}$$

$$+\left(\frac{\rho h}{k}\right)^2\frac{\partial^4}{\partial\tau^4}\left\{\frac{h^2}{12}\cdot\frac{5-\gamma}{1-\gamma}\nabla\nabla w+\frac{2}{1-\gamma}\frac{1}{k}\Phi+\right.$$
$$+\frac{2}{R^2(1-\gamma)}\cdot\left(1+3\frac{h\varepsilon^2R^2}{\rho h}\right)w-\frac{2}{1-\gamma}\frac{\chi_2}{k}-$$
$$\left.-\frac{3-\gamma}{1-\gamma}\left(1+2\frac{\varepsilon^2h^2}{12\rho h(3-\gamma)}\right)\cdot\nabla w\right\}+\left(\frac{\rho h}{k}\right)^2\frac{\varepsilon h}{k}\frac{\partial^5}{\partial\tau^5}\times \qquad (5.52)\ \text{cont.}$$
$$\times\left\{-\frac{6}{1-\gamma}w+\frac{4h^2}{12(1-\gamma)}\nabla w\right\}+\left(\frac{\rho h}{k}\right)^3\frac{\partial^6}{\partial\tau^6}\times$$
$$\times\left\{\frac{2}{1-\gamma}w-\frac{h^2}{12}\cdot\frac{2}{1-\gamma}\nabla w\right\}=0.$$

Equation (5.52) in dimensionless coordinates when $\varepsilon = 0$ has the form

$$\frac{\partial^2}{\partial\tau^2}\left\{w-\frac{h^2}{12R^2}\nabla \underline{w}\right\}+\frac{\partial^4}{\partial\tau^4}\left\{\frac{3-\gamma}{2}\underline{\nabla}\left(\frac{h^2}{12R^2}\cdot\frac{5-\gamma}{3-\gamma}\nabla w-w\right)+\right.$$
$$\left.+\underline{w}+\frac{1}{k}\underline{\Phi^*}\right\}+\frac{1-\gamma}{2}\frac{\partial^2}{\partial\tau^2}\left\{\nabla\nabla w-2\frac{2-\gamma}{1-\gamma}\frac{h^2}{12R^2}\underline{\nabla\nabla\nabla} w-\right. \qquad (5.53)$$
$$\left.-\frac{3-\gamma}{1-\gamma}\nabla\Phi\frac{1}{k}-\frac{\partial^2 w}{\partial\beta^2}-(3+2\gamma)\frac{\partial^2 w}{\partial\alpha^2}\right\}+\frac{1-\gamma}{2}S=0,$$

where

$$\chi=\alpha R,\quad S=\beta R,$$
$$S=\frac{h^2}{12R^2}\nabla\nabla\nabla\nabla w+\frac{1}{k}\nabla\nabla\Phi^*+(1-\gamma^2)\frac{\partial^4 w}{\partial\alpha^4}, \qquad (5.54)$$
$$\nabla(\ )=\frac{\partial^2(\ )}{\partial\alpha^2}+\frac{\partial^2(\ )}{\partial\beta^2}.$$

Underlined terms express the effect of tangent inertial forces of rotation. Tangential forces of inertia transmit their effect on transverse vibrations mainly through the terms containing tangential inertial forces separately, we find (5.53) when $\frac{\partial^2 u}{\partial t^2} = 0$ and $\frac{\partial^2 v}{\partial t^2} = 0$ respectively,

$$\frac{\partial^4}{\partial\tau^4}\left\{\frac{\partial^2 u_0}{\partial\alpha^2}+\frac{1-\gamma}{2}\frac{\partial^2 v_0}{\partial\beta^2}\right\}+\frac{1-\gamma}{2}\frac{\partial^2}{\partial\tau^2}\left\{\nabla\nabla\left[w-\right.\right.$$
$$\left.-\frac{3-\gamma}{1-\gamma}\cdot\frac{h^2}{12R^2}\frac{\partial^2 w}{\partial\alpha^2}-2\frac{h^2}{12R^2}\frac{\partial^2 w}{\partial\beta^2}\right]-\frac{2}{1-\gamma}\cdot\frac{\partial^2}{\partial\alpha^2}\cdot\frac{\Phi^*}{k}-$$
$$\left.-\frac{1}{k}\frac{\partial^2}{\partial\beta^2}\Phi^*-2(1+\gamma)\frac{\partial^2 w}{\partial\alpha^2}-\frac{\partial^2 w}{\partial\alpha^2}-\frac{\partial^2 w}{\partial\beta^2}\right\}+\frac{1-\gamma}{2}S=0, \qquad (5.55)$$

$$\frac{\partial^4}{\partial\tau^4}\left\{\frac{1-\gamma}{2}\frac{\partial^2\chi_0}{\partial\alpha^2}+\frac{\partial^2\chi_0}{\partial\beta^2}\right\}+\frac{1-\gamma}{2}\frac{\partial^2}{\partial\tau^2}\left\{\nabla\nabla\left(w-\frac{k^2}{6R^2}\frac{\partial^2 w}{\partial\alpha^2}-\right.\right.$$

$$\left.\left.-\frac{3-\gamma}{1-\gamma}\frac{k^2}{12R^2}\frac{\partial^2 w}{\partial\beta^2}\right)-\frac{1}{k}\frac{\partial^2\Phi^*}{\partial\alpha^2}-\frac{2}{1-\gamma}\frac{1}{k}\frac{\partial^2\Phi^*}{\partial\beta^2}-\frac{\partial^2 w}{\partial\alpha^2}\right\}+$$

$$+\frac{1-\gamma}{2}S=0, \qquad (5.56)$$

$$\Phi^*=N\frac{\partial^2 w}{\partial\alpha^2}+2T\frac{\partial^2 w}{\partial\alpha\,\partial\beta}+pR\frac{\partial^2 w}{\partial\beta^2}, \quad \chi_0=\frac{k^2}{12R^2}\Delta w-w.$$

We estimate the terms, containing inertial forces of rotation, assuming that in time only the amplitude changes, whereas the number of transverse and longitudinal waves does not change. Let us assume that $r^2 = n^2 + m^2\frac{\pi^2 R^2}{l^2}$, where n is the number of transverse waves, m — number of longitudinal waves. For average lengths $r \sim (m^2 + n^2)^{1/2}$. The effect of inertial forces of rotation of the order of h/R, if $r \sim (R/h)^{1/3}$, of the order of $(h/R)^{1/2}$, if $r \sim (R/h)^{3/4}$, and of the order of 1, if $r \sim R/h$.

We consider the case, when $r \lesssim (R/h)^{1/2}$. Then, inertial forces of rotation may be disregarded. Since we use the motion equation, where we have disregarded the shift as compared to unity, then we have already disregarded the values of the order of inertial forces of rotation.

For thick shells the effect of inertial forces of rotation may be significant.

Thus, disregarding the effect of inertial forces of rotation in equation (5.53), we obtain,

$$\frac{\partial^6 w}{\partial\tau^6}+\frac{\partial^4}{\partial\tau^4}\left\{w-\frac{3-\gamma}{2}\nabla w+\frac{1}{k}\Phi^*\right\}+\frac{1-\gamma}{2}\frac{\partial^2}{\partial\tau^2}\times$$

$$\times\left\{\nabla\nabla w-\frac{3-\gamma}{1-\gamma}\cdot\frac{1}{k}\nabla\Phi^*-\frac{\partial^2 w}{\partial\beta^2}-(3+2\gamma)\frac{\partial^2 w}{\partial\alpha^2}\right\}+$$

$$+\frac{1-\gamma}{2}S=0. \qquad (5.57)$$

The equation of such type when $\Phi^* = 0$ was obtained by V. Flugge [53].

The equation (5.56) automatically includes the static stability equation [47]

$$S = 0. \qquad (5.58)$$

Disregarding tangential forces of inertia, we obtain,

$$\frac{\partial^2}{\partial \tau^2}\nabla\nabla w + \varepsilon^* \frac{\partial}{\partial \tau}\nabla\nabla w + S - \frac{1}{R}\nabla\nabla x_3 = 0, \qquad (5.59)$$

where

$$\varepsilon^* = -\varepsilon R\left(\frac{k}{\rho k}\right)^{1/2}.$$

Let us consider the natural vibration of the cylinder of average length with hinged fastening of ends. We select the sag in the form

$$w = \sum_{m,n} f_{n,m}(\tau)\sin\lambda_m \alpha \cdot \sin\beta_n. \qquad (5.60)$$

Choosing $f_{n,m} = A_{n,m}\begin{cases}\sin\omega_{nm}\tau \\ \cos\omega_{nm}\tau\end{cases}$ and substituting expression (5.60) in (5.57), we find when $\Phi^* = 0$,

$$\omega^6_{nm} - \varphi_4\omega^4_{mn} + \varphi_2\omega^2_{mn} - \varphi_0 = 0, \qquad (5.61)$$

where $\omega^2_{mn}$ is the immeasurable angular frequency,

$$\varphi_4 = 1 + \frac{3-\gamma}{2}r^2, \quad \varphi_2 = \frac{1-\gamma}{2}\left[r^4 + n^2(3+2\gamma)\lambda^2_m\right],$$
$$\varphi_0 = \frac{1-\gamma}{2}\left[\frac{D}{kR^2}r^8 + (1-\gamma^2)\lambda^4_m\right]. \qquad (5.62)$$

Determining $\omega_{mn}$ from equation (5.61), we find,

$$\Omega^2_{nm} = \frac{1}{R^2}\cdot\frac{k}{\rho(1-\gamma^2)}\omega^2_{mn}. \qquad (5.63)$$

Designating $\omega^2_{mn} = z$, for determining z we obtain the cubic equation

$$z^3 - \varphi_4 z^2 + \varphi_2 z - \varphi_0 = 0. \qquad (5.64)$$

First we will consider transverse oscillations without calculation of tangential forces of inertia. From (5.59) when $\varepsilon = 0$ and $x_1 = 0$ we obtain,

$$\omega^2_{mn} = \frac{h^2}{12R^2}r^4 + (1-\gamma^2)\frac{\lambda^4_m}{r^4}. \qquad (5.65)$$

Extreme values $\omega^2_{mn}$ are found from conditions,

a) m = 1, and n from condition b) $\frac{h^2}{12R^2} r^8 = (1 - \gamma^2)\lambda^4_m$. Condition "b" designates that with a certain oscillation frequency the energy of bend should be equal to the energy of extension of the middle surface. Taking "b" into consideration, the expression (5.65) can be rewritten in the form,

$$\omega^2_{mn\,\text{наим}} = \left(\frac{1-\gamma^2}{3}\right)^{1/2} \frac{\pi^2 Rh}{l^2};$$
$$\Omega_{mn\,\text{наим}} = \left(\frac{1-\gamma^2}{3}\right)^{1/4} \left(\frac{\pi}{l}\right)^2 \frac{k}{\rho R}. \tag{5.66}$$

[наим = min = minimum]

Formula (5.66) is very convenient for practical calculations. We find roots (5.64). Let us assume that $z_1 > z_2 > z_3$, where $z_1$ and $z_2$ express frequencies of natural tangential oscillations. Let us note that $z_1$ and $z_3$ could be obtained with a sufficient accuracy from (5.55), and $z_2$ and $z_3$ – from (5.56). Comparing $z_3$ and $\omega^2$ from (5.65) when m = 1, R = 20 cm, h = 0.1, $\gamma$ = 0.3, we are convinced that the effect of tangential inertial forces decreases rapidly with the growth of n. In determining the least natural frequencies we can disregard the effect of tangential forces of inertia. But they have a strong effect when there are oscillations with a small number of transverse waves. For instance, when n = 1 tangential forces of inertia decrease the transverse frequency to 40-45%, when n = 2 – 12-16%, and when n = 3 – 6-7%. It would seem that with small n the equations of sloping shells cannot be used. However, for a circular cylinder of average length, with small n the term of bending character $(\lambda^2 + h^2)^4 \frac{h^2}{12R^2}$ is considerably less than $(1 - \gamma^2)\lambda^4$. If we take into consideration the intersepting forces in the first two motion

equations and changes of curvatures at the expense of tangential displacement, then we shall obtain an additional expression of the same order or less than $(\lambda^2 + n^2)^4 \frac{h^2}{12R^2}$, but, of course, considerably less than the term $(1 - \gamma^2)\lambda^4$ (when n are small).

Problems on the effect of preliminary load on natural oscillation frequencies do not present special difficulties. For instance, it is easy to find such an N, with which the cylindrical shell would have the required natural oscillation frequencies.

Let us now assume that the closed circular cylindrical shell is subjected to the action of hydrostatic pressure rapidly increasing in time.* Let us assume that ends of the shell are hinged to circular frames. Let us assume, as before, that frames can be deformed in their own plane, remaining circular.

We assume that the external-pressure q, which acts on the shell, changes in proportion to time t

$$q = ct. \tag{5.67}$$

Let us use the differential equations of the nonlinear theory of flexible shells. In the equation of equilibrium of the shell element we introduce an additional term, considering the force of inertia and corresponding to sag w. We shall not consider the forces of inertia, corresponding to displacements in the middle surface.

In other words, here we do not examine the phenomena of propagation of elastic waves in the middle surface of the shell.

Let us also assume that the shape of the shell is not ideal and that its middle surface has certain initial dents comparable with thickness.

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*The solution of this problem belongs to V. E. Mineyev [69].

In order to find the dependency between parameters of sag and the load variable in time we use the Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+Q_j=0, \tag{5.68}$$

where T is the kinetic energy of the system, $q_j$ is the generalized coordinate, and $Q_j$ is the generalized force. For generalized coordinates we select parameters of sag. Whether tolerances admitted are taken into consideration, the differential equations finally attain the form,

$$\begin{aligned}\frac{D}{h}\nabla^2\nabla^2 w={}&\frac{\partial^2(w+w_{\text{нч}})}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2(w+w_{\text{нч}})}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2}-\\&-2\frac{\partial^2(w+w_{\text{нч}})}{\partial x\,\partial y}\frac{\partial^2\varphi}{\partial x\,\partial y}+\frac{1}{R}\frac{\partial^2\varphi}{\partial x^2}+\frac{q}{h}-\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2},\end{aligned} \tag{5.69}$$

$$\begin{aligned}\frac{1}{E}\nabla^2\nabla^2\varphi={}&\left[\frac{\partial^2(w+w_{\text{нч}})}{\partial x\,\partial y}\right]^2-\frac{\partial^2(w+w_{\text{нч}})}{\partial x^2}\frac{\partial^2(w+w_{\text{нч}})}{\partial y^2}-\\&-\left[\left(\frac{\partial^2 w_{\text{нч}}}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w_{\text{нч}}}{\partial x^2}\frac{\partial^2 w_{\text{нч}}}{\partial y^2}\right]-\frac{1}{R}\frac{\partial^2 w}{\partial x^2},\end{aligned} \tag{5.70}$$

where $\gamma$ is the specific gravity of the shells material, g is acceleration due to gravity, and t is time.

The expression for the function of initial sag we, as before, take in the form

$$w_{\text{нч}}=f_{\text{нч}}(\sin\alpha x\sin\beta y+\psi\sin^2\alpha x+\chi) \tag{5.71}$$

and for the additional sag

$$W=f(\sin\alpha x\sin\beta y+\psi\sin^2\alpha x+\chi),$$
$$\text{where }\alpha=\frac{\pi}{L},\quad\beta=\frac{n}{R}. \tag{5.72}$$

Further, integrating equation (5.70), we find the function of stresses in the middle surface,

$$\begin{aligned}\frac{1}{E}\varphi={}&r_1\cos 2\alpha x+r_2\cos 2\beta y+r_3\sin\alpha x\sin\beta y+\\&+r_4\sin\alpha x\sin\beta y-\frac{qR}{2Eh}x^2-\frac{qR}{4Eh}y^2.\end{aligned} \tag{5.73}$$

Coefficients $r_1$, $r_2$, $r_3$, and $r_4$ are determined by the corresponding expressions (4.8).

Determine the total energy of the system

$$Э = \hat{v}_1 + \hat{v}_2 - w_1 - \hat{w}_2, \tag{5.74}$$

which will be written in the form

$$\begin{aligned} Э = {} & \frac{1}{2} c_1 \xi^2 (\xi + 2\xi_{нач})^2 + \frac{1}{2} c_2 (\xi + 2\xi_{нач}) \zeta^2 - \\ & - c_3 (\xi + 2\xi_{нач}) \xi\zeta + \frac{1}{2} c_4 \xi^2 + \frac{1}{2} c_5 \zeta^2 - c_6 \hat{q} \xi (\xi + 2\xi_{нач}) - \\ & - \frac{1}{4} c_7 \hat{q} \xi (\xi + 2\xi_{нач}) - \frac{1}{2} c_7 \hat{q} \frac{\zeta^2}{\xi} (\xi + 2\xi_{нач}) + \frac{1}{2} c_8 \hat{q}_s. \end{aligned} \tag{5.75}$$

Coefficients $c_i = 1, \ldots, 8$ are determined according to (4.25). We find derivatives from the energy of the system by parameters of sag $\zeta$ and $\xi$:

$$\begin{aligned} \frac{\partial Э}{\partial \xi} = {} & 2c_1 \xi (\xi + \xi_{нач})(\xi + 2\xi_{нач}) + c_2 (\xi + 2\xi_{нач}) \zeta^2 - \\ & - 2c_3 (\xi + \xi_{нач}) \zeta + c_4 \xi - 2c_6 \hat{q} (\xi + \xi_{нач}) - \frac{1}{2} c_7 \hat{q} (\xi + \xi_{нач}) = 0, \end{aligned} \tag{5.76}$$

$$\begin{aligned} \frac{\partial Э}{\partial \zeta} = {} & c_2 (\xi + 2\xi_{нач}) \zeta - c_3 \xi (\xi + 2\xi_{нач}) + \\ & + c_5 \zeta - c_7 \hat{q} \frac{\zeta}{\xi} (\xi + \xi_{нач}) = 0. \end{aligned} \tag{5.77}$$

The kinetic energy T of the system is equal to:

$$T = \frac{1}{2} \int_0 \int_0^{2\pi R} \frac{\gamma h}{g} \left( \frac{\partial w}{\partial t} \right)^2 dx\, dy. \tag{5.78}$$

The expression of additional sag (5.72) we present in dimensionless values according to dependencies

$$\hat{w} = \xi \sin \alpha x \sin \beta y + \zeta \sin^2 \alpha x + \lambda. \tag{5.79}$$

Further we use the condition of closure (4.18), expressions (4.20) and define from it the parameter of sag,

$$\lambda = \frac{1}{2} \hat{q} (2 - \nu) + \frac{1}{2} c_8 (\xi^2 + 2\xi\xi_{нач}) - \frac{1}{2} \zeta. \tag{5.80}$$

Let us introduce expressions (5.80) and (5.79) and write the derivative $\hat{w}$ with respect to time,

$$\begin{aligned} \frac{\partial \hat{w}}{\partial t} = {} & \dot{\xi} \sin \alpha x \sin \beta y + \dot{\zeta} \sin^2 \alpha x + c_8 \dot{\xi} (\xi + \xi_{нач}) - \\ & - \frac{1}{2} \dot{\zeta} + \frac{1}{2} (2 - \nu) \dot{\hat{q}}. \end{aligned} \tag{5.81}$$

We present (5.78) in dimensionless values, introducing the designation

$$\hat{T} = \frac{R}{\pi E I h^3} T. \tag{5.82}$$

We now transform equation (5.78), substituting in it expressions (5.81) and (5.82). Integrating, we obtain,

$$\begin{aligned}\hat{T} = c_9 \Big[ & \frac{1}{2}\dot{\xi}^2 + \frac{1}{4}\dot{\zeta}^2 + 2c_6^2(\xi + \xi_{нч})^2\dot{\xi}^2 + \\ & + \frac{1}{2}(2-\nu)^2\dot{\hat{q}}^2 + 2\dot{\xi}\dot{\hat{q}}(\xi + \xi_{нч})(2-\nu)c_6 \Big],\end{aligned} \tag{5.83}$$

where through $c_9$ we designated

$$c_9 = \frac{1}{2}\frac{\gamma}{g}\frac{R^2}{E}. \tag{5.84}$$

We obtain the first Lagrange equation by putting in equation (5.68) expressions (5.76), (5.81) and considering that

$$\hat{q}_j = \xi; \quad \hat{Q}_j = \frac{\partial Э}{\partial \xi},$$

then we have

$$\begin{aligned} & c_9[1 + 4c_6^2(\xi + \xi_{нч})^2]\ddot{\xi} + 4c_9c_6^2(\xi + \xi_{нч})\dot{\xi}^2 + \\ & + 2c_9c_6(2-\nu)(\xi + \xi_{нч})\ddot{\hat{q}} + 2c_1\xi(\xi + \xi_{нч})(\xi + 2\xi_{нч}) + \\ & + c_3(\xi + 2\xi_{нч})\zeta^2 - 2c_3(\xi + \xi_{нч})\zeta + c_4\xi - \\ & - 2c_6\hat{q}(\xi + \xi_{нч}) - \frac{1}{2}c_7\hat{q}(\xi + \xi_{нч}) = 0. \end{aligned} \tag{5.85}$$

The second Lagrange equation for

$$\hat{q}_j = \zeta; \quad \hat{Q}_j = \frac{\partial Э}{\partial \zeta}$$

now has the form

$$\begin{aligned} & \frac{1}{2}c_9\ddot{\zeta} + c_3\xi(\xi + 2\xi_{нч})^2\zeta - c_3\xi^2(\xi + 2\xi_{нч}) + \\ & + c_5\xi\zeta - c_7\hat{q}(\xi + \xi_{нч}) = 0. \end{aligned} \tag{5.86}$$

We substitute in equation (5.85) values of coefficients $c_i$ from (4.25) and (5.84) and, assuming $\ddot{\hat{q}} = 0$, we obtain,

$$\hat{q}=\frac{(1+\vartheta^4)\eta_1}{16\left(1+\frac{\vartheta^2}{2}\right)}\xi(\xi+2\xi_{\text{нач}})+\frac{\eta_1}{\left(1+\frac{\vartheta^2}{2}\right)}\left[\frac{\vartheta^4}{(1+\vartheta^2)^2}+\right.$$

$$\left.+\frac{\vartheta^4}{(1+9\vartheta^2)^2}\right]\frac{\xi+2\xi_{\text{нач}}}{\xi+\xi_{\text{нач}}}\zeta^2-\frac{1}{4\left(1+\frac{\vartheta^2}{2}\right)}\left[1+\frac{8\vartheta^4}{(1+\vartheta^2)^2}\right]\zeta+$$

$$+\frac{1}{\left(1+\frac{\vartheta^2}{2}\right)}\left[\frac{\vartheta^4}{(1+\vartheta^2)^2\eta_1}+\frac{\eta_1(1+\vartheta^2)^2}{12(1-\nu^2)}\right]\frac{\xi}{\xi+\xi_{\text{нач}}}+$$

$$+\frac{\gamma R^2}{4gE}\frac{1}{\eta_1\left(1+\frac{\vartheta^2}{2}\right)}[4+\eta_1^2(\xi+\xi_{\text{нач}})^2]\frac{1}{\xi+\xi_{\text{нач}}}+\frac{d^2\xi}{dt^2}+$$

$$+\frac{\gamma R^2}{4gE}\frac{\eta_1}{\left(1+\frac{\vartheta^2}{2}\right)}\left(\frac{d\xi}{dt}\right)^2. \tag{5.87}$$

Further, we introduce the dimensionless time parameter $\hat{t}$

$$\hat{t}=\frac{ctR^2}{Eh^2\hat{q}_h}, \tag{5.88}$$

where through $\hat{q}_h$ we designate expression (4.31).

Considering the value of dimensionless parameter of time, we transform the expression (5.87),

$$\hat{t}=\frac{(1+\vartheta^4)\eta_1}{16\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_h}\xi(\xi+2\xi_{\text{нач}})+\frac{\eta_1}{\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_h}\times$$

$$\times\left[\frac{\vartheta^4}{(1+\vartheta^2)^2}+\frac{\vartheta^4}{(1+9\vartheta^2)^2}\right]\frac{\xi+2\xi_{\text{нач}}}{\xi+\xi_{\text{нач}}}\zeta^2-\frac{1}{4\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_h}\times \tag{5.89}$$

$$\times\left[1+\frac{8\vartheta^4}{(1+\vartheta^2)^2}\right]\zeta+\frac{\xi}{\xi+\xi_{\text{нач}}}+\frac{1}{4\eta_1\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_h^2}\times$$

$$\times\frac{\gamma R^4c^2}{gE^2h^4}[4+\eta_1^2(\xi+\xi_{\text{нач}})^2]\times$$

$$\times\frac{1}{\xi+\xi_{\text{нач}}}\frac{d^2\xi}{d\hat{t}^2}+\frac{\eta_1}{4\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_h^2}\frac{\gamma R^4c^2}{gE^2h^4}\left(\frac{d\xi}{d\hat{t}}\right)^2.$$

Now, we introduce designation

$$S=\left(1+\frac{\vartheta^2}{2}\right)\eta_1\hat{q}_h^2\left(\frac{VE}{cR}\right)^2\left(\frac{R}{h}\right)^4, \tag{5.90}$$

where $V=\sqrt{\frac{gE}{\gamma}}$ is the velocity of propagation of sound in the shell

material.

The first Lagrange equation now has the final form

$$\frac{d^2\xi}{d\hat{t}^2}+\eta_1^2 N\left(\frac{d\xi}{d\hat{t}}\right)^2-S\left\{4N\left(\hat{t}-\frac{\xi}{\xi+\xi_{\text{нч}}}\right)-\frac{(1+\vartheta^4)\eta_1}{4\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_0}N\xi\times\right.$$
$$\times(\xi+2\xi_{\text{нч}})-\frac{4\eta_1}{\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_0}\,\frac{\gamma R^4c^2}{gE^2h^4}\left[4+\eta_1^2(\xi+\xi_{\text{нч}})^2\right]\frac{1}{\xi+\xi_{\text{нч}}}\times$$
$$\times\frac{d^2\xi}{d\hat{t}^2}+\frac{\eta_1}{4\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_0^3}\,\frac{\gamma R^4c^2}{gE^2h^4}\left(\frac{d\xi}{d\hat{t}}\right)^2. \tag{5.91}$$

where

$$N=\frac{\xi+\xi_{\text{нч}}}{4+\eta_1^2(\xi+\xi_{\text{нч}})^2}.$$

Performing analogous transformations with the second Lagrange equation (5.86), we have

$$\frac{d^2\zeta}{d\hat{t}^2}-\frac{1}{\eta_1\left(1+\frac{\vartheta^2}{2}\right)\hat{q}_0}S\left\{\hat{t}\vartheta^2\eta_1(\xi+\xi_{\text{нч}})\hat{q}_0-2\left[\frac{1}{4}+\frac{\eta_1^2\vartheta^4}{S(1-\mu^2)}\right]\zeta-\right.$$
$$-\vartheta^4\eta_1^2\left[\frac{1}{(1+\vartheta^2)^2}+\frac{1}{(1+9\vartheta^2)^2}\right](\xi+2\xi_{\text{нч}})\xi\cdot\zeta+$$
$$\left.+\eta_1\left[\frac{1}{S}+\frac{\vartheta^4}{(1+\vartheta^2)^2}\right](\xi+\xi_{\text{нч}})\xi^2\right\}=0. \tag{5.92}$$

Thus, we have arrived at the system of ordinary nonlinear second-order differential equations (5.91) and (5.92), which connect parameters of sag $\xi$, $\zeta$, load $\hat{q}$ and time $\hat{t}$.

Let us adduce the results of numerical integration. In the approximate solution of the problem integration of the system of differential equations (5.91) and (5.92) will be replaced by integration of the first equation only (5.91). Parameter of sag $\zeta$ we determine by the equation, satisfying the solution of static problem.

We take the following initial conditions,

$$\frac{d\xi}{d\hat{t}}=0,\ \xi=0 \text{ when } \hat{t}=0. \tag{5.93}$$

In calculations we assume: R = 9 cm; R/h = 112.5; L/R = 2.2; E = $7.75\cdot10^5$ kg/cm$^2$; $\nu$ = 0.3; V = $5\cdot10^5$ cm/sec, $\xi_{in}$ = 0.001.

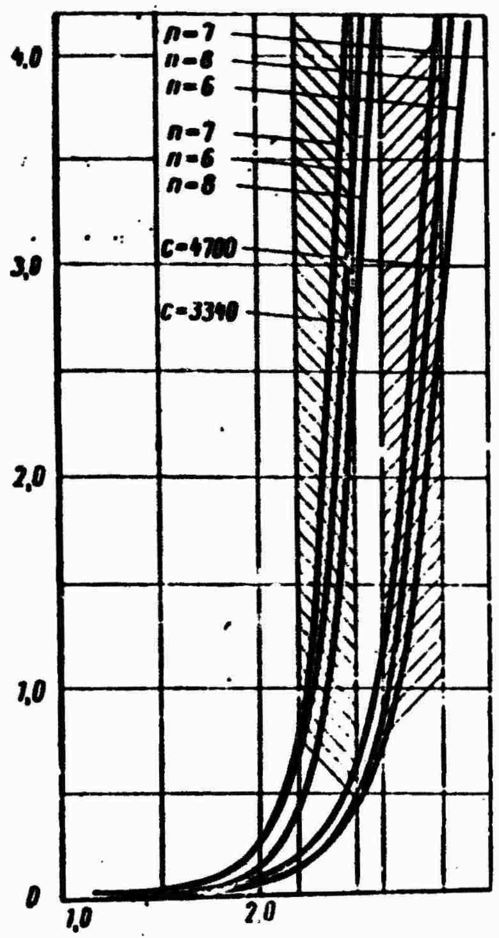


Fig. 50.

Results of calculations of the rates of growth of load C = 3340 and C = 4700 atmospheres/sec for different values n are presented in Fig. 50.

In conclusion, we will note the very interesting recent research of V. V. Bolotin, pertaining to the problem of shell dynamics. He has shown that if the spectrum of random forces is sufficiently wide, then simultaneously oscillation are excited in degrees of freedom, and, consequently, the question on density of natural frequencies of the plate and shell is of interest. Corresponding estimates for plates, as we know, were obtained by Courant; for shells, the oscillation of which are described by equations for states with large index of variability, analogous estimates were given by Bolotin.* The latter scientist** has also suggested a method for the study of the behavior of plates and shells with random loads which uses essentially the plurality of the excited degrees of freedom. Under specific, sufficiently broad conditions integral estimates for correlation functions and spectral densities of generalized coordinates were obtained. The use of the

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*PMM. No. 2, Vol. 25, 1963.

**News of Higher Educational Institutions, "Machine Building," No. 3, 1963.

theory of dynamic boundary "effect" enabled him to calculate average squares of stresses, appearing near lines of distortion in the shell, and to investigate their dependency on parameters, of the problem. Bolotin has also studied the problem of limitations, which have to be superimposed on shell properties and load properties so that the stationary distribution of dynamic variables of the system was described by Maxwell-Boltzman* distribution. He showed that delta correlation of the load in time is not a sufficient condition. The load should be delta correlated on the middle surface; moreover, certain limitations are imposed on damping forces.

*Symposium "Problems of Dynamics and Dynamic Strength," No. 7, publication of the Academy of Sciences, Latvian SSR, 1963.

## § 6. Statistical Method of Investigation of the Stability of Shells

The problem of stability of shells, namely, the determination of its forms of equilibrium — is connected with the solution of equations of the nonlinear theory of shells. Depending upon the value of parameter $\lambda$ of the load a certain form of equilibrium is possible. However, even if we managed to solve with accuracy nonlinear equations of the theory of shells, even in that case the problem cannot be considered to be thoroughly investigated, since there remains a vague degree of reality of each form of equilibrium of the shell which is possible when $\lambda_0 < \lambda < \lambda_2$.

For selection of the most real form of the shell's equilibrium we should introduce in our examination certain additional considerations. I. I. Vorovich [70] considers it rational to take the measure of reality of a certain form of shell equilibrium the probability of the shell staying in this form.

The application of the probability theory to the research on shells will allow us to advance the solution of problems on the establishment of permissible loads on the shell during research on stability, taking into account conditions of its work and errors in manufacture; on the establishment of tolerances in the fulfillment of basic shell parameters.

Let us consider the approximate approach to setting up of the statistical theory of shell stability, offered by I. I. Vorovich. Let us divide all factors, determining the random character of the flexure of shell, in three groups: 1) scattering of elastic and geometric properties of the shell; 2) scattering of parameters, characterizing methods of sealing the shell; 3) scattering of external loads applied to the shell.

Further, although in the groups shown functional parameters can also be included, as for example, an aberration of the shape of the middle surface of the shell, digression in the thickness of the shell etc., nevertheless we will assume that all the totality of factors of the first two groups may be described by the finite number of parameters $a_1, \ldots, a_m$. Therefore, it is natural to consider that probability properties of the first two groups of factors will be given, if the law $\varphi(a_1, \ldots, a_m)$ of distribution of parameters $a_1, \ldots, a_m$ is given. Let us assume now that parameters $a_1, \ldots, a_m$ are fixed, and write equations of the motion of the shell under the action of load $F(P, t)$ taking into account the dissipation of energy during motion of the shell. We have:

$$\rho \frac{\partial^2 w}{\partial t^2} + 2\gamma \frac{dw}{dt} + D\nabla^4 w = \frac{\partial^2 \varphi}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}\right) + \frac{\partial^2 \varphi}{\partial x^2} \times \times\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 f}{\partial y^2}\right) - 2\frac{\partial^2 \varphi}{\partial x\,\partial y}\left(\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial x\,\partial y}\right) + z(p, t), \tag{6.1}$$

$$\nabla^4 \varphi = 2Eh\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 f}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\,\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}\right]. \tag{6.2}$$

In these equations $\rho$ is the mass density of shell, referred to one unit of the area of middle surface of the shell; the dispersion of energy in the shell is taken into consideration by term $2\gamma \frac{\partial w}{\partial t}$. For simplicity, in equations (6.1) and (6.2) we disregarded the inertia of longitudinal motions of shells and considered that $\overline{F}(P, t)$ has only one component $Z(P, t)$. All these assumptions can be discarded although this will bring about certain complication of subsequent computations.

We assume that for w uniform conditions of support are fulfilled and, furthermore,

$$\varphi/r = r(s), \quad \frac{\partial \varphi}{\partial n} = q(s), \tag{6.3}$$

where r(s), q(s) are certain functions of contour s.

We search for the approximate solution of the problem in the following form:

$$w = \sum_{k=1}^{n} q_k(t) f_k(p). \tag{6.4}$$

Here $f_k(p)$ is the basis in the space of energy of the flexure of shell. For determination of $q_k(t)$ we use Bubnov-Galerkin's method, assuming that $f_k$ are orthonormal in $L_B$. Now we obtain the following system,

$$\ddot{q}_k + \frac{2\gamma}{\rho}\dot{q}_k = -\frac{1}{\rho}\frac{\partial v}{\partial q_k} + \frac{1}{\rho} z_k(t) \; (k = 1, \ldots, n) \quad \left(z = \int_{\Omega} z(p, t) f_k(p)\, dp\right). \tag{6.5}$$

Here v is the potential strain energy of the shell, expressed through $q_k$.

System (6.5) may be considered as the equation of motion of a certain point in n-dimensional space of coefficients $q_1, \ldots, q_n$. This point of motion is in the field of forces with potential $\rho^{-1}v$ and under the action of random forces $\rho^{-1}z_k(t)$. From now on we will consider that

$$z(p, t) = z^{(1)}(p, t) - z^{(2)}(p, t) + z^{(3)}(p, t) \quad (z^{(1)}(p, t) = M.OZ(p, t)). \tag{6.6}$$

Here $z^{(2)}(p, t)$ is the fluctuation term, producing the acceleration of point of the type of Brownian motion accelerations $z^{(3)}(p, t)$ is a continuous random process.

We shall assume further that with a sufficient degree of accuracy we can write

$$z^{(3)}(p, t) = \sum_{k=1}^{n} \sum_{l=1}^{n_1} a_{kl} f_k(p) \psi_l(t). \tag{6.7}$$

Here $\psi_l(t)$ are certain fixed time functions. Let us consider that the continuous random process is given, if the law of distribution $\Theta(a_{kl})$ of parameters $a_{kl}$ is known. In accordance with (6.6) we have:

$$z_k(t) = z_k^{(1)}(t) + z_k^{(2)} + \sum_{s=1}^{n_1} a_{ks}\psi_s(t). \tag{6.8}$$

The problem now consists in finding the law of time distribution of $q_1, \ldots, q_n$.

For its solution we will assume that groups of parameters $a_1, \ldots, a_m$ and $a_{kl}$ and the random process $z^{(2)}(p, t)$ are statistically independent. Let us assume further that parameters $a_1, \ldots, a_m, a_{kl}$ received a certain fixed value, and now we find the law of distribution of $q_1, \ldots, q_n$ in this assumption. If we assume that $z^{(2)}$ is white noise δ-correlated on the middle surface in time, then for time moments $t \gg \rho/\gamma$ the distributive law sought can be found from Smolukhovskiy equation:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left\{\left[\frac{\partial v}{\partial q_i} - z_i^{(1)}(t) - \sum_{s=1}^{n_1} a_{is}\psi_s\right] f\right\}\frac{1}{2\gamma} + \frac{\delta^2}{4\gamma^2}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial q_i^2}. \tag{6.9}$$

In equation (6.9) parameter δ characterizes scattering of impacts acting on the shell, and the smaller the δ, the less the scattering of impacts acting on the shell. The parameter characterizes conditions, under which the shell works, and should be determined from experiment.

Inasmuch as f is a certain distributive law, then to (6.9) we must add the following conditions, taking place when $t > 0$:

$$1)\ f > 0; \qquad 2)\ \int_{-\infty}^{+\infty}\cdots\int f\,dq_1, \ldots, dq_n = 1;$$
$$3)\ f \to 0 \quad \text{as} \quad q_1^2 + \ldots + q_n \to \infty. \tag{6.10}$$

Furthermore, $f(q_1, \ldots, q_n, 0) = f^*(q_1, \ldots, q_n)$, where $f^*$ is the law of distribution of $q_1, \ldots, q_n$ in the initial moment.

Let us assume that we managed to find f from (6.9), (6.10). Obviously, f will also depend on parameters $a_1, \ldots, a_m, a_{kl}$; also,

the absolute law of distribution of $f^0$ for the conditions considered will be

$$f^0=\int_{-\infty}^{+\infty}\cdots\int f(q_1,\ldots,q_n,t,a_k,a_{kl})\,\varphi(a_k)\,\theta(a_{kl})\,da_k\,da_{kl}. \tag{6.11}$$

Let us consider certain important cases, when a full realization of the above-stated plan is possible and when calculation formulas can be obtained.

Let us assume that $z^{(3)} \equiv 0$, and $z^{(1)}$ does not depend on time. In this case the distribution of $f(q_1, \ldots, q_n)$, which will be established when $t \to \infty$, will be determined from equation

$$\frac{k\psi^2}{2\gamma}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial q_i^2}+\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left[\left(\frac{\partial v}{\partial q_i}-z_i^{(1)}\right)f\right]=0. \tag{6.12}$$

We can easily check that function

$$\frac{1}{I}\exp\left[\left(-v+\sum_{k=1}^{n}q_k z_k^{(1)}\right)\frac{2\gamma}{k\psi^2}\right],$$
$$I=\int_{-\infty}^{+\infty}\cdots\int\exp\left[\left(-v+\sum_{k=1}^{n}q_k z_k^{(1)}\right)\frac{2\gamma}{k\psi^2}\right]dq_1\ldots dq_n \tag{6.13}$$

satisfies all conditions (6.10) and the equation (6.9). Distribution (6.13) is Gibbs distribution.

The absolute law of distribution in accordance with (6.13) was determined by formula

$$f^0(q_1,\ldots,q_n)=\int_{-\infty}^{+\infty}\cdots\int f(q_1,\ldots,q_n,a_1\ldots a_n a_{kl})\,\varphi(a_k)\times$$
$$\times\,\theta(a_{kl})\cdot da_k\,da_{kl}. \tag{6.14}$$

The value $f^0$ may be taken for the measure of reality of a certain form of equilibrium of the shell.

Formula (6.14) gives a sufficiently full solution.

The condition of $\delta$-correlation of the process $z^{(2)}$, taken in work [70], enabled us to obtain a closed solution in the form (6.13), (6.14). If we discard this condition, then the distributive law for

f from (6.9), and (6.10) will have to be found numerically.

Let us note essential features of the recommended method of statistical analysis of the equilibrium of shells.

1. Calculation by formula (6.14) does not require a preliminary solution of the problem on the shell equilibrium, an analysis of the number of forms of equilibrium, or a replacement of real dependencies between sags and external forces, by single-valued functions, etc. It is only required to know the expression of potential energy of the system through generalized coordinates.

2. The calculation of the distributive law by formula (6.14) is reduced to taking of quadratures. Inasmuch as integrands in formula (6.14) are sufficiently smooth functions, these quadratures without any complications can be taken numerically, even if for the increase of accuracy of solution of the problem we resort to the use of a large number of parameters $q_1, \ldots, q_n$. Here, naturally, no special questions arise, connected with the use of machines for calculation by formula (6.14).

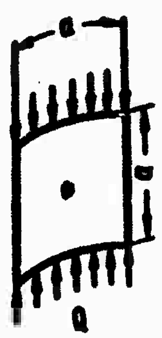

Fig. 51.

3. Formula (6.14) in principle considers all basic factors, determining the random character of bend of the shell, including such factors, as random forces, which change in time very rapidly, and forces with the period of change, comparable with the period of oscillations of the shell proper, etc. Moreover, it makes it possible to trace the process of change of probabilities in time. It is true, a resolution of the corresponding boundary-value problem for equation (6.9) will be also required here.

But equation (6.9) belongs to the number of those equations for the solution of which numerical methods are very suitable.

We consider the stability of a square cylindrical panel under the action of longitudinal compressing force Q (Fig. 51). In solving the problem we take into consideration probable deviations in the shape of the middle surface and the effect of random fast-changing external forces.

The potential energy of the shell can be taken in the form [4]:

$$
\begin{gathered}
v=\frac{\pi^4 E h^4}{8a^2}\left\{(\zeta^4+4\zeta^3\zeta_0+4\zeta_0^2\zeta^2)-\frac{64k}{\pi^4}\left(\frac{5\zeta^3}{9}+\zeta^2\zeta_0\right)+\right.\\
\left.+\frac{16}{\pi^2}\zeta^2(S_B-S)-\frac{32}{\pi^2}S\zeta\zeta_0\right\}, \qquad (6.15)\\
\zeta=\frac{f}{2h},\quad \zeta_0=\frac{f_0}{2h},\quad k=\frac{a^2}{2R\cdot h},\quad S_B=3{,}6+\frac{k^2}{30{,}5},\\
S=\frac{Qa^2}{4Eh^3}.
\end{gathered}
$$

Here a is the side of the shell square, 2h is the thickness of shell, E is Young's modulus, f is the sag of shell, $f_0$ is the initial sag.

We consider the shell with curvature parameters k = 12. In this case for the potential energy we obtain the formula

$$
\begin{gathered}
v=\frac{\pi^4 E\cdot h^4}{8a^2}\{\zeta^4+\zeta^3(4\zeta_0-4{,}36)+\zeta^2[-3{,}86\cdot\zeta_0+4{,}85(1-p)]-\\
-2{,}341\zeta\zeta_0 p\},\quad P=S/S_B. \qquad (6.16)
\end{gathered}
$$

In accordance with (6.13) conditional distributive law $\zeta$ (the distributive law for the determined $\zeta_0$) is yielded by relationship

$$
\begin{gathered}
f(\zeta,\zeta_0)=\frac{1}{I}e^{-\mu V(\zeta)},\quad I=\int_{-\infty}^{\infty}e^{-\mu V(\zeta)}d\zeta, \qquad (6.17)\\
\mu=\frac{\pi^4 E h^4 \gamma}{4a^2\rho^2\varkappa}.
\end{gathered}
$$

In equalities (6.17) we introduce the designation

$$
\begin{gathered}
V(\zeta)=\zeta^4+\zeta^3(4\zeta_0)-4{,}36+\zeta^2[-3{,}86\zeta_0+11{,}85(1-p)]-\\
-23{,}41\zeta\zeta_0 p.
\end{gathered}
$$

The absolute distributive law will be given by formula

$$
f^{\circ}(\zeta)=\int_{-\infty}^{\infty}f(\zeta,\zeta_0)\varphi(\zeta_0)d\zeta_0. \qquad (6.18)
$$

where $\varphi(\zeta_0)$ is the law of distribution of $\zeta_0$.

Let us determine, for instance, with the help of (6.18) the probability that the modulo displacement of $\zeta$ with respect to the modulus will not exceed one unity. Obviously,

$$p=\int_{-1}^{+1} f^{*}(\zeta_{*})\,d\zeta_{*}=\int_{-1}^{+1}\int_{-\infty}^{\infty} f(\zeta_{*},\zeta_{0})\,\varphi(\zeta_{0})\,d\zeta_{0}\,d\zeta_{*}. \tag{6.19}$$

Results of numerical calculations by the formulas are given in Figs. 52 and 53, for the case, when $\zeta_0$ is subordinate to the triangular symmetric distributive law.

In Fig. 52 $p(D\zeta_0)$ is plotted, where $D\zeta_0$ is the dispersion of $\zeta_0$. Calculations were performed for the case when $p = 0.5$ (i.e., for the case, when the compressing force constitutes half of the upper critical value) and for $\mu = 1; 0.5; 0.2; 0.1$. Parameter $\mu$ for a fixed shell depends on $\delta$ the value, characterizing the working conditions of the shell. The larger the $\delta$, the "calmer" the working conditions of the shell. From Fig. 52 it is clear that with a sufficiently small $\mu$, i.e., under not very "calm" working conditions of the shell, $D\zeta_0$ practically has no effect on p.

In Fig. 53 $p(D\zeta_0)$ is plotted when $\mu = 1$ and with different p. From Fig. 53 it is clear that $p(D\zeta_0)$ has a different character for different p. If $p < 0.544$ is the lower critical number for the given case, then an increase of $D\zeta_0$ results in a decrease at p. If, however, $p > 0.544$, then with an increase of $D\zeta_0$ the value p increases also. This circumstance paradoxical at the first glance, is fully explainable.

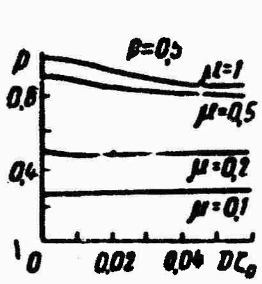


Fig. 52

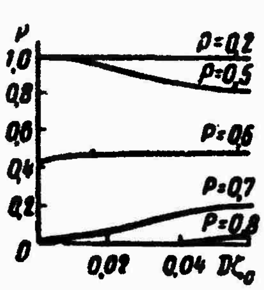


Fig. 53.

Indeed, the detailed analysis of a number of forms of equilibrium of the shell and the degree of their stability shows that when $p > p_0$ ($p_0$ is the lower critical load) with large positive $\zeta_0$ there exists only one form of equilibrium, to which correspond $\zeta$, lying outside $[-1, +1]$.

With small positive $\zeta_0$ the shell has three forms of equilibrium, and one of these forms lies inside the segment $[-1, +1]$.

However, to this form corresponds a higher level of potential energy of the shell than forms, lying outside $[-1, +1]$. Therefore, although with small positive $\zeta_0$ there are forms of equilibrium inside $[-1, +1]$, they give us little for increasing the probability of realization of inequality $|\zeta| < 1$. However, with negative $\zeta_0$ there are also positions of equilibrium to which correspond $\zeta$ from segment $[-1, +1]$. However, here for negative $\zeta_0$ precisely these forms appear to be the stablest, and the larger the $\zeta_0$, the stabler the corresponding form. Therefore, when we decrease dispersion of $\zeta_0$, decreasing thereby the probability of appearance of sufficiently large negative $\zeta_0$, the probability p can decrease.

If $p < 0.544$, then to every $\zeta_0$ corresponds the only form of equilibrium of the shell, and the smaller the $\zeta_0$, the less the value of $\zeta$, corresponding to the form of equilibrium of the shell. Here, of course, with the decrease of dispersion of $\zeta_0$ the value of p should be increased. We shall also note that if we decrease $D\zeta_0$, concentrating the distributive law for $\zeta_0$ on negative $\zeta_0$, then we will always have an increase of p. Therefore, it is natural to pose the question of introduction of measures of technological, constructive and other order, by means of which we could create artificial dispersion, concentrating the law of distribution of $\zeta_0$ on negative values.

In Fig. 54 we plot p against P for different $D\zeta_0$. We can note that function p(P) experiences a sharp change when values of the load are somewhat larger than the lower critical number. These load values are characterized by the circumstance that three forms of equilibrium of shell correspond to it, and in two stable forms of equilibrium the shell has equal levels of potential energy.

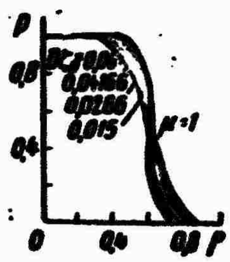

Fig. 54.

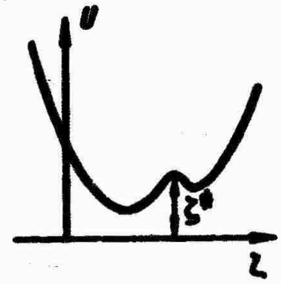

Fig. 55.

Graphs in Fig. 54 are plotted for $\mu = 1$ and, consequently, can be used only when the working conditions of the shell correspond to $\mu = 1$. However, it is quite possible to plot a series of such graphs for various $\mu$. This would enable us by the given level of probability of the shell staying in a certain state under given working conditions, to determine the permissible scattering in the form of the shell's middle surface. Figure 55 depicts the curve of the potential energy of the shell – external forces system. With a certain value of $p > 0.544$ the pop of shell will occur, if under the action of random impacts the potential barrier $\zeta_*$ will be surmounted. Therefore, it is possible to assume approximately that for a fixed $\zeta_0$ the probability of pop $p_*$ will be given by the relationship

$$p_* = \int_{\zeta_*}^{\infty} f(\zeta, \zeta_0)\, d\zeta. \tag{6.20}$$

Using the theorem of full probability, we obtain the following formula for calculation of the probability of popping:

$$p_{1*} = \int_{-\infty}^{\infty} p_*(\zeta_0)\, \varphi(\zeta_0)\, d\zeta_0. \tag{6.21}$$

Further, if we take into consideration that popping can take place only when $\zeta_0$, satisfies the inequality

$$\zeta_0 < \zeta_{0*}(P), \tag{6.22}$$

where $\zeta_{0*}$ is a certain number specific for every P, then formula (6.21) can be written in the form

$$P_{1*} = \int_{-\infty}^{\zeta_{0*}} \int_{\zeta}^{\infty} f(\zeta, \zeta_0)\,\varphi(\zeta_0)\, d\zeta\, d\zeta_0 \qquad (6.23)$$

Results of calculations by the formula (6.23) are shown in Fig. 56. Here we can also note the circumstance that with the increase of $D\zeta_0$ the probability of popping decreases. This is explained by the fact that by decreasing $D\zeta_0$, we render improbable large modulo values of $\zeta_0$ (we remind the reader that the distributive law was assumed to be symmetric). But with large positive $\zeta_0$ popping does not occur at all, and for large negative $\zeta_0$ popping is unlikely, since the pre-popping state of equilibrium in this case has a lower energy level than the post-popping state.

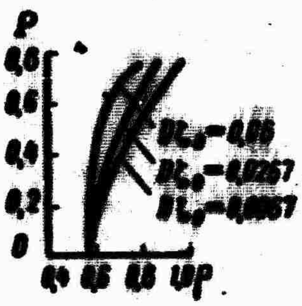

Fig. 56.

In conclusion let us note that in using the above-stated method, it will evidently be necessary to divide all possible real conditions of exploitation of shells into calculation instances according to the level of "calmness" of the work, and for every calculation instance to establish $\mu$ experimentally.

# CHAPTER VI

## STABILITY OF SHELLS BEYOND THE ELASTIC LIMIT

### § 1. Formulation of the Problem

Phenomena of instability are characterized by the fact that with certain values of external forces, along with given (zero-moment) state of equilibrium of shell other states of equilibrium are found to be possible also.

Let us investigate stability in the case of elastoplastic state of material of a shell.

Following A. A. Il'yushin [2], let us examine a deformed state of the shell infinitely close to the given state, and characterized by elongations $e_{xx} + \delta e_{xx}$, $e_{yy} + \delta e_{yy}$ and shift $e_{xy} + \delta e_{xy}$ in layer ABC (Fig. 57), located at a distance z from the middle surface; variations of stress $\delta X_x$, $\delta Y_y$, $\delta X_y$, which can be calculated on the basis of laws of plasticity correspond to deformation variations $\delta e_{xx}$, $\delta e_{yy}$, $\delta e_{xy}$. Inasmuch as in this case we speak of real variations of deformations, and not about virtual ones, as in the variational equation of equilibrium, it is necessary to distinguish two possible cases; the case of loading and the case of unloading, inasmuch as formulas, connecting stressed and deformed states are different here.

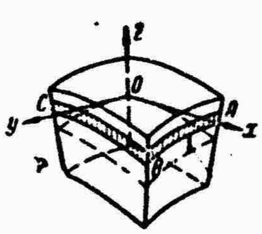

Fig. 57.

The loading region is characterized by the fact that in it at the expense of variations $\delta e_{xx}$, ..., $\delta X_x$ the intensity of deformations and stresses increases; in the region of unloading however, these values decrease. The surface, intersecting the thickness of the shell and separating the regions of loading and unloading, is determined, consequently, from the condition of equality to zero either of the variation of intensity of deformations or intensity of stresses. In view of the fact that the variation of the work of internal forces in one unit volume of the shell is equal to

$$\sigma_i \delta e_i = X_x \delta e_{xx} + Y_y \delta e_{yy} + X_y \delta e_{xy}, \tag{1.1}$$

i.e., is proportional to $\delta e_i$, then the equation of the above surface will be

$$X_x \delta e_{xx} + Y_y \delta e_{yy} + X_y \delta e_{xy} = 0. \tag{1.2}$$

It may be obtained directly by using a variation of formula

$$e_i = \frac{2}{\sqrt{3}} \sqrt{e_{xx}^2 + e_{xx}e_{yy} + e_{yy}^2 + \frac{1}{4} e_{xy}^2} \tag{1.3}$$

and simple transformations according to:

$$\begin{aligned} S_x &= X_x - \frac{1}{2} Y_y = \frac{\sigma_i}{e_i} e_{xx}, \\ S_y &= Y_y - \frac{1}{2} X_x = \frac{\sigma_i}{e_i} e_{yy}, \\ S_{xy} &= X_y = \frac{\sigma_i}{3e_i} e_{xy} \end{aligned} \tag{1.4}$$

and

$$\sigma_i = \sqrt{X_x^2 - X_x Y_y + Y_y^2 + 3X_y^2}. \tag{1.5}$$

In the loading region variations of stresses can be found by means of differentiation of formulas (1.4), since they take place both in the principal and similar states of the shell:

$$\delta S_x \equiv \delta X_x - \frac{1}{2} \delta Y_y = \frac{\sigma_i}{e_i} \delta e_{xx} + e_{xx} \frac{d}{de_i}\left(\frac{\sigma_i}{e_i}\right) \delta e_i, \quad \delta S_y \equiv \delta Y_y - \frac{1}{2} \delta X_x = \frac{\sigma_i}{e_i} \delta e_{yy} + e_{yy} \frac{d}{de_i}\left(\frac{\sigma_i}{e_i}\right) \delta e_i,$$

$$\delta S_{xy} \equiv \delta X_y = \frac{\sigma_i}{3e_i} \delta e_{xy} + \frac{1}{3} e_{xy} \frac{d}{de_i}\left(\frac{\sigma_i}{e_i}\right) \delta e_i. \tag{1.6}$$

where $\sigma_i$ and $e_i$ are connected by a diagram of extension $\sigma_i = \Phi(e_i)$, so that

$$\frac{d}{de_i}\left(\frac{\sigma_i}{e_i}\right) = -\frac{1}{e_i}\left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right) < 0. \tag{1.7}$$

In the unloading region variations of stresses and deformations obey the Hook's law, and the connection between them can be found from (1.6), if we assume that $\sigma_i = Ee_i$:

$$\delta S_x = E\delta e_{xx}, \quad \delta S_y = E\delta e_{yy}, \quad \delta X_y = \frac{1}{3} E\delta e_{xy}. \tag{1.8}$$

As we did in the general theory of shells, we will proceed from the basic Kirchhoff's hypothesis, namely; we assume that variations of deformation of a layer of shell ABC (Fig. 57) are expressed by linear relationships through variations of deformations of the middle surface and through its distortions:

$$\delta e_{xx} = \varepsilon_1 - z\varkappa_1, \quad \delta e_{yy} = \varepsilon_2 - z\varkappa_2, \quad \delta e_{xy} = 2(\varepsilon_3 - z\varkappa_3), \tag{1.9}$$

while here with $\varepsilon_1$, $\varepsilon_2$. $2\varepsilon_3$, we designate infinitesimal variations of deformations of the middle surface, and with $\varkappa_1$, $\varkappa_2$, $\varkappa_3 = \tau$ – infinitesimal variations of its curvatures and torsion.

For convenience of calculation we will introduce designation of dimensionless values; a line above the value of stress will be used to note the relation of this stress to the intensity of stresses $\sigma_i$:

$$\frac{X_x}{\sigma_i} = \overline{X}_x, \quad \frac{Y_y}{\sigma_i} = \overline{Y}_y, \quad \frac{X_y}{\sigma_i} = \overline{X}_y, \quad \frac{S_x}{\sigma_i} = \overline{S}_x, \quad \frac{S_y}{\sigma_i} = \overline{S}_y. \tag{1.10}$$

These values are known; instead of distortions $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ and ordinate $z$ we will introduce dimensionless values:

$$\frac{h}{2}\varkappa_1 = \overline{\varkappa}_1, \quad \frac{h}{2}\varkappa_2 = \overline{\varkappa}_2, \quad \frac{h}{2}\varkappa_3 = \overline{\varkappa}_3, \quad \frac{2z}{h} = \overline{z}. \tag{1.11}$$

Let us now write formula (1.1) in the form

$$\delta e_i = \varepsilon - z\varkappa = \varepsilon - \bar{z}\bar{\varkappa}, \tag{1.12}$$

where

$$\begin{gathered} \varepsilon = \bar{X}_x \varepsilon_1 + \bar{Y}_y \varepsilon_2 + 2\bar{X}_y \varepsilon_3, \\ \varkappa = \bar{X}_x \varkappa_1 + \bar{Y}_y \varkappa_2 + 2\bar{X}_y \varkappa_3, \\ \bar{\varkappa} = \frac{h}{2}\varkappa. \end{gathered} \tag{1.13}$$

If we now use $z_0$ to designate the ordinate of the surface, separating the regions of loading and unloading, then according to (1.2) and (1.12) we will obtain,

$$z_0 = \frac{\varepsilon}{\varkappa}, \quad \bar{z}_0 = \frac{\varepsilon}{\bar{\varkappa}}. \tag{1.14}$$

To be specific we will assume that the loading region adjoins the external surface of the shell $z = +\frac{h}{2}$. In this case for $z > z_0$ there are formulas (1.6), where, according to the adopted designations of dimensionless values, they can be rewritten in the form:

$$\begin{gathered} \delta S_x = \left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)\bar{S}_x \bar{\varkappa}(\bar{z} - \bar{z}_0) + \frac{\sigma_i}{e_i}(\varepsilon_1 - \bar{\varkappa}_1 \bar{z}), \\ \delta S_y = \left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)\bar{S}_y \bar{\varkappa}(\bar{z} - \bar{z}_0) + \frac{\sigma_i}{e_i}(\varepsilon_2 - \bar{\varkappa}_2 \bar{z}), \\ \delta X_y = \left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)\bar{X}_y \bar{\varkappa}(\bar{z} - \bar{z}_0) + \frac{2\sigma_i}{3e_i}(\varepsilon_3 - \bar{\varkappa}_3 \bar{z}). \end{gathered} \tag{1.15}$$

Formulas (1.8), taking place in the unloading region $z < z_0$, will be written thus:

$$\begin{gathered} \delta S_x = E(\varepsilon_1 - \bar{\varkappa}_1 \bar{z}), \\ \delta S_y = E(\varepsilon_2 - \bar{\varkappa}_2 \bar{z}), \\ \delta X_y = \frac{2}{3}E(\varepsilon_3 - \bar{\varkappa}_3 \bar{z}). \end{gathered} \tag{1.16}$$

As we see from the comparison of formulas (1.15) and (1.16), on the boundary of loading and unloading regions ($z = z_0$) variations of stresses, in general, are not continuous functions of z. Their infinitesimal jumps are proportional to difference $E - \frac{\sigma_i}{e_i}$, i.e.,

they disappear, if the material of the shell exceeds only very little the limits of elasticity with respect to the value of intensity of deformation $e_i$ (here, obviously, $\frac{d\sigma_i}{de_i}$ may be as small as desired). The discontinuity of stresses disappears also, when the variation of the stressed state is simple, i.e., if the variations of stresses are proportional to the active stresses; in this case on the boundary of regions of loading and unloading they will turn into zero together with the variation of intensity of stresses $\delta\sigma_i$ (or intensity of deformations $\delta e_i$), since they will be proportional to $\delta\sigma_i$.

Thus, speaking in principle, the discontinuity of values $\delta X_x$ on boundary $z = z_0$ will take place in those cases, when the loss of stability of shell is accompanied by complicated loading of elements of the material, i.e., either the discontinuity or continuity of stresses may be established after the problem on the stability of the shell is solved. Hence it is clear that the degree of accuracy of the solution of the problem of stability of shells, accuracy meaning the degree of conformity of mathematical solution with experimental data, will be fully sufficient, if values of jumps of stress variations on the $z = z_0$ boundary will be small in comparison with variations of stresses on the shell's surface $\left(z = \pm\frac{h}{2}\right)$. otherwise, an experimental check of solutions is necessary. The difficulty, which we encounter here, is inevitable not only within the framework of the theory of small elastoplastic deformations, but also from the point of view of any other theory of plasticity. Let us note that intermittent change of stress variations during transition over the boundary $z = z_0$ is the inevitable result of continuity of deformations, their intensity and the intensity of stresses, inasmuch as transformation of material from the plastic state into the elastic

state with the constant intensity of stresses is connected with redistribution of stresses.

Formulas (1.15), (1.16) show that stress variations are linear functions of ordinate z, while, in contrast to the case of elastic loss of stability, they depend not only on deformations and mechanical characteristics of the shell's material, but also on stresses acting prior to the loss of stability, and consequently on forces. This constitutes the specific feature of the phenomenon of loss of stability of shell beyond the limits of elasticity.

In order to be able to write the differential equations of stability, it is necessary to find the expression for variations of forces and moments, acting on the element of the shell, inasmuch as they ensure from equations of equilibrium of the element.

For determination of forces and moments we have:

$$\delta T_1 - \frac{1}{2}\delta T_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta S_x\,dz; \quad \delta T_2 - \frac{1}{2}\delta T_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta S_y\,dz;$$

$$\delta S = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_y\,dz; \quad \delta M_1 - \frac{1}{2}\delta M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta S_x z\,dz; \tag{1.17}$$

$$\delta M_2 - \frac{1}{2}\delta M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta S_y z\,dz; \quad \delta H = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_y z\,dz.$$

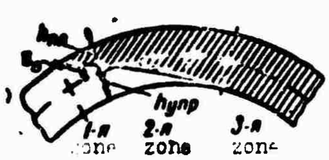

Fig. 58.
[пл = pl = plastic; упр = el = = elastic]

For calculation of these integrals first of all it is necessary to divide the shell into the following three zones: in the first zone (Fig. 58) the shell is assumed to be in the elastic state,* and

*If prior to loss of stability in a certain zone of the shell it is in the elastic state, then with infinitesimal variations, generally speaking, it will remain elastic.

therefore, according to (1.16), (1.17) for this zone we have,

$$\frac{1}{Eh}\left(\delta T_1 - \frac{1}{2}\delta T_2\right) = \varepsilon_1; \quad \frac{1}{Eh}\left(\delta T_2 - \frac{1}{2}\delta T_1\right) = \varepsilon_2;$$
$$\frac{1}{Eh}\delta S = \frac{2}{3}\varepsilon_3; \tag{1.18}$$
$$\frac{4}{3D}\left(\delta M_1 - \frac{1}{2}\delta M_2\right) = -\varkappa_1, \quad \frac{4}{3D}\left(\delta M_2 - \frac{1}{2}\delta M_1\right) = -\varkappa_2,$$
$$\frac{4}{3D}\delta H = -\frac{2}{3}\varkappa_3.$$

Here, the cylindrical rigidity is designated with D, as before

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{1.19}$$

The second zone is characterized by the fact that prior to loss of stability the shell material in it was deformed plastically, and after the loss of stability a part of the layer converts into the elastic state, i.e., in this zone there are both a region of active plastic deformation (loading), and a region of unloading. Each of the integrals (1.17) in this zone should be divided into two parts, from $z = -\frac{h}{2}$ to $z = z_0$ and from $z = z_0$ to $z = +\frac{h}{2}$, where the first one should be calculated according to the formula (1.16), and the second one – according to (1.15), for instance:

$$\delta T_1 - \frac{1}{2}\delta T_2 = \frac{hE}{2}\int_{-1}^{\bar{z}_0}(e_1 - \bar{\varkappa}_1 z)\,d\bar{z} + \frac{h}{2}\left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)\bar{S}_x\bar{\varkappa}\int_{\bar{z}_0}^{1}(\bar{z} - \bar{z}_0)\,d\bar{z} +$$
$$+ \frac{h}{2}\frac{\sigma_i}{e_i}\int_{\bar{z}_0}^{1}(\varepsilon_1 - \bar{\varkappa}_1\bar{z})\,d\bar{z}.$$

Thus, during calculation we encounter the simplest integrals of the type:

$$\int d\bar{z} = \bar{z}, \quad \int \bar{z}\,d\bar{z} = \frac{1}{2}\bar{z}^2, \quad \int \bar{z}^2\,d\bar{z} = \frac{1}{3}\bar{z}^3.$$

Let us use the adopted designations of the known values:

$$\omega = 1 - \frac{1}{E}\frac{\sigma_i}{e_i}, \quad \lambda = 1 - \frac{1}{E}\frac{d\sigma_i}{\partial e_i}. \tag{1.20}$$

Then, calculating the first group of integrals (1.17), we will obtain expressions for variations of forces:

$$\frac{4}{Eh}\left(\delta T_1-\frac{1}{2}\delta T_2\right)=2(1-\omega+\omega\bar{z}_0)\,\varepsilon_1+\omega(1-\bar{z}_0^2)\,\bar{\varkappa}_1+$$
$$+(\lambda-\omega)\bar{S}_x(1-\bar{z}_0)^2\bar{\varkappa},$$
$$\frac{4}{Eh}\left(\delta T_1-\frac{1}{2}\delta T_2\right)=2(2-\omega+\omega\bar{z}_0)\,\varepsilon_2+\omega(1-\bar{z}_0^2)\,\bar{\varkappa}_2+$$
$$+(\lambda-\omega)\bar{S}_y(1-\bar{z}_0)^2\varkappa. \tag{1.21}$$
$$\frac{12}{Eh}\delta S=4(2-\omega+\omega\bar{z}_0)\,\varepsilon_3+2\omega(1-\bar{z}_0^2)\,\bar{\varkappa}_3+$$
$$+3(\lambda-\omega)\bar{X}_y(1-\bar{z}_0)^2\varkappa.$$

After calculation of the second group of integrals we find the formulas for variations of moments:

$$\frac{16}{3D}\left(\delta M_1-\frac{1}{2}\delta M_2\right)=2(2-\omega+\omega\bar{z}_0^3)\,\varkappa_1+$$
$$+(\lambda-\omega)(1-\bar{z}_0)^2(2+\bar{z}_0)\bar{S}_x\varkappa-\frac{6\omega}{h}(1-\bar{z}_0^2)\,\varepsilon_1,$$
$$\frac{16}{3D}\left(\delta M_2-\frac{1}{2}\delta M_1\right)=-2(2-\omega+\omega\bar{z}_0^3)\,\varkappa_2+$$
$$+(\lambda-\omega)(1-\bar{z}_0)^2(2+\bar{z}_0)\bar{S}_y\varkappa-\frac{6\omega}{h}(1-\bar{z}_0^2)\,\varepsilon_2,$$
$$\frac{16}{3D}\delta H=-4(2-\omega+\omega\bar{z}_0^3)\,\varkappa_3+ \tag{1.22}$$
$$+3(\lambda-\omega)(1-\bar{z}_0)^2(2+\bar{z}_0)\bar{X}_y\varkappa-\frac{12\omega}{h}(1-\bar{z}_0^2)\,\varepsilon_3.$$

In the third zone of the shell deformation of the shell which was plastic until the loss of stability remains plastic also after the loss of stability, i.e., the region of unloading is absent. Therefore, expressions of variations of forces and moments are obtained from (1.17) according to formulas (1.15),

$$\frac{1}{Eh}\left(\delta T_1-\frac{1}{2}\delta T_2\right)=(1-\omega)\,\varepsilon_1-(\lambda-\omega)\bar{S}_x\varepsilon,$$
$$\frac{1}{Eh}\left(\delta T_2-\frac{1}{2}\delta T_1\right)=(1-\omega)\,\varepsilon_2-(\lambda-\omega)\bar{S}_y\varepsilon, \tag{1.23}$$
$$\frac{1}{Bh}\delta S=\frac{2}{3}(1-\omega)\,\varepsilon_3-(\lambda-\omega)\bar{X}_y\varepsilon,$$
$$\frac{4}{3D}\left(\delta M_1-\frac{1}{2}\delta M_2\right)=-(1-\omega)\,\varkappa_1+(\lambda-\omega)\bar{S}_x\varkappa, \tag{1.24}$$

$$\frac{4}{3D}\left(\delta M_2 - \frac{1}{2}\delta M_1\right) = -(1-\omega)\varkappa_2 + (\lambda-\omega)\bar{S}_y\varkappa,$$ (1.24) con'd

$$\frac{4}{3D}\delta H = -\frac{2}{3}(1-\omega)\tau_3 + (\lambda-\omega)\bar{X}_y\varkappa.$$

Formulas (1.18), (1.23), and (1.24) for the first and third zones establish linear uniform relations between variations of forces and moments, on the one hand, and deformation of the middle surface and its distortions – on the other hand. However, in the zone of elastoplastic deformations (second) these relationships are not linear, while they remain uniform. This can be seen from formulas (1.21), and (1.22), which include value $\bar{z}_0$, which is a linear-fractional function of the zero degree with respect to $\varepsilon_n$ and $\varkappa_n$:

$$\bar{z}_0 = \frac{\bar{X}_x\varepsilon_1 + \bar{Y}_y\varepsilon_2 + 2\bar{X}_y\varepsilon_3}{\bar{X}_x\bar{\varkappa}_1 + \bar{Y}_y\bar{\varkappa}_2 + 2\bar{X}_y\bar{\varkappa}_3}. \quad (1.14')$$

It is very essential that in this expression at $\bar{z}_0$ we can exclude deformations $\varepsilon_n$, expressing them through variations of forces $\delta T_1$, $\delta T_2$, and $\delta S$. Multiplying the first equation of group (1.21) by $\bar{X}_x$, the second by $\bar{Y}_y$ and the third by $\bar{X}_y$ and adding them up, we see that deformations $\varepsilon_n$ are included in the obtained equation only in the form of certain combinations of $\varepsilon$; but since from (1.14) $\varepsilon = \bar{z}_0\bar{\varkappa}$, then, excluding this value, we will obtain:

$$\lambda(1-\bar{z}_0)^2 + 4\bar{z}_0 - 4\frac{\bar{S}_x\delta T_1 + \bar{S}_y\delta T_2 + 3\bar{X}_y\delta S}{Eh\bar{\varkappa}} = 0. \quad (1.25)$$

Let us designate with $\varphi$ the dimentionless value, included in this equation and depending on variations of forces and curvatures:

$$\varphi = \frac{\lambda}{1-\lambda}\,\frac{\bar{S}_x\delta T_1 + \bar{S}_y\delta T_2 + 3\bar{X}_y\delta S}{Eh\bar{\varkappa}}. \quad (1.26)$$

Solving quadratic equation (1.25), we find,

$$\zeta = \frac{1-\sqrt{(1-\lambda)(1+\varphi)}}{\lambda}, \quad (1.27)$$

where $\zeta$ is the relation of the thickness of the plastic layer of the shell to its total thickness (Fig. 58):

$$\zeta = \frac{1-\bar{z}_0}{2} = \frac{h_{n\Lambda}}{h}, \quad \bar{z}_0 = 1 - 2\zeta. \tag{1.28}$$

Thus, in formulas (1.21) and (.122) $\bar{z}_0$ means either the expression of this value (1.14') through deformations, its expression through variations of forces and distortions (1.28).

Expressions of forces and moments in the zone of elastoplastic deformations of shell are somewhat simplfied, if before the loss of stability the plastic deformation is small in comparison with the elastic deformation. Rejecting in (1.21) and (1.22) small values of the order of $\omega$ in comparison with 1 and replacing $\bar{z}_0$ according to formula (1.28), we will obtain:

$$\begin{aligned} \frac{1}{Eh}\left(\delta T_1 - \frac{1}{2}\delta T_2\right) &= \varepsilon_1 + \frac{\lambda h}{2}\bar{S}_x \varkappa \zeta^2; \\ \frac{1}{Eh}\left(\delta T_2 - \frac{1}{2}\delta T_1\right) &= \varepsilon_2 + \frac{\lambda h}{2}\bar{S}_y \varkappa \zeta^2; \\ \frac{1}{Eh}\delta S &= \frac{2}{3}\varepsilon_3 + \frac{\lambda h}{2}\bar{X}_y \varkappa \zeta^2; \end{aligned} \tag{1.29}$$

$$\begin{aligned} \frac{4}{3D}\left(\delta M_1 - \frac{1}{2}\delta M_2\right) &= -\varkappa_1 + \lambda \bar{S}_x \zeta^2 (3 - 2\zeta)\varkappa, \\ \frac{4}{3D}\left(\delta M_2 - \frac{1}{2}\delta M_1\right) &= -\varkappa_2 + \lambda \bar{S}_y \zeta^2 (3 - 2\zeta)\varkappa, \\ \frac{4}{3D}\delta H &= -\frac{2}{3}\varkappa_3 + \lambda \bar{X}_y \zeta^2 (3 - 2\zeta)\varkappa. \end{aligned} \tag{1.30}$$

Fundamental simplification of basic relationships (1.21) and (1.22) occurs in those cases, when from certain considerations value $\zeta$, i.e., the relative thickness of the plastic layer in the second zone, can be considered a known function of coordinates of a point of the surface. Actually, here the relationships indicated, as well as relationships (1.23) and (1.24), become linear and uniform with respect to force factors of deformations and distortions, and

therefore, the problem on the stability of shells beyond the limit of elasticity in the mathematical sense will be somewhat more complicated than the corresponding elastic problem.

We will not write out here the differential equations of equilibrium of the element of a shell of arbitrary shape, inasmuch as they do not differ in any way from equations, adopted in the theory of elastic stability of shells, and we will limit ourselves to certain remarks only. In the general instance this system of five first-order differential equations on forces $\delta T_1$, $\delta T_2$, $\delta S$, moments $\delta M_1$, $\delta M_2$, $\delta H$ and serving forces $\delta N_1$, $\delta N_2$; the first three equations are obtained from the condition of equilibrium of projections of forces $\delta T_1$, $\delta T_2$, $\delta S$, $\delta N_1$, $\delta N_2$, on directions of x, y, z axes of the basic trihedron (Fig. 57); the last two equations are equations of equilibrium of moments of forces with respect to x, y axes. In view of the fact that components of deformation $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ and distortions $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are expressed according to the known Love's formulas with three components of displacement of the point of the middle surface $u(\alpha, \beta)$, $v(\alpha, \beta)$, $w(\alpha, \beta)$, the above-derived formulas (1.18), (1.21), (1.22), (1.23) and (1.24) allow us to express values $\delta T_1$, $\delta T_2$, $\delta S$, and $\delta M_1$, $\delta M_2$ $\delta H$ with u, v, w, and therefore, five equations of equilibrium will contain five unknown functions: $\delta N_1$, $\delta N_2$, u, v, w. To them we must add boundary conditions, of which static boundary conditions are reduced to the fact that variations of external forces on the boundary of the shell are equal to zero, inasmuch as the loss of stability of the shell should occur in the presence of constant external forces. Another formulation of the problem of stability consists of the fact that on the basis of relationships of type (1.21), (1.22) and expression of $\varepsilon_n$, $\varkappa_n$, through u, v, w we set up differential equations of compatibility of deformations, expressed

through force factors $\delta T_1$, $\delta T_2$, $\delta S$, $\delta M_1$, $\delta M_2$, $\delta H$; to these equations we add five equations of equilibrium and boundary conditions. Finally, the third formulation of the problem consists of application of the variational equation of equilibrium of the theorem of the minimum of energy.

## § 2. Closed Cylindrical Shell

Let us examine the cylindrical form of the loss of stability of a cylindrical shell, compressed with external pressure q and axial force P [2]. Let us select axes of coordinates x, y, as it was shown in Fig. 59. In view of the fact that external forces are constant along the x axis and the shell is a circular cylindrical one, stresses in it everywhere are constant and equal:

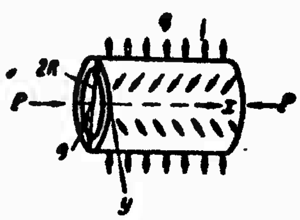

Fig. 59.

$$Y_y = -q\frac{R}{h}, \quad X_x = -\frac{P}{2\pi Rh}, \quad X_y = 0. \tag{2.1}$$

The problem on the stability of such a shell may be solved exactly: sag w is a function of angle θ only, and therefore,

$$\varkappa_1 = \varkappa_3 = 0, \qquad \varkappa_2 = -\frac{1}{R^2}\left(\frac{d^2w}{d\theta^2} + w\right).$$

Furthermore, from the equation of equilibrium of variations of forces, acting on the element along x axis, and condition of constancy of the stressed state along the x axis it follows that

$$\delta T_1 = \delta S = 0.$$

Actually, in view of the fact that the shell in the direction of x is assumed to be sufficiently long, its cross section always remains planar, and therefore, shift $\varepsilon_3$ is absent; from the third equation of group (1.21) we have $\delta S = 0$. Consequently, the equation of equilibrium of forces in the direction of x axis has the form

$$\frac{\delta T_1}{dx} = 0, \qquad \delta T_1 = 0.$$

Further computations are simple. However, they are significantly simplified in one particular case, when force P is equal

$$P = \pi R^2 q,$$

i.e., when the very same uniform pressure as on the lateral surface act on shell bottoms. In this case

$$Y_y = 2X_x, \quad S_x = X_x - \frac{1}{2}Y_y = 0, \tag{2.3}$$

i.e., the strain before the loss of stability is plane, and therefore, it will remain plane after the loss of stability also, consequently, the elongation of $\varepsilon_1$ will be equal to zero ($\varepsilon_1 = 0$). From the first equation of group (1.21) we have here:

$$\delta T_2 = 0,$$

and from the second equation we can find deformation of $\varepsilon_2$, which, however, subsequently will be needed no longer. Equations (1.22) will be transformed to form:

$$\begin{aligned} \frac{\delta M_1}{D} &= -(1-\psi)\left(\varkappa_1 + \frac{1}{2}\varkappa_2\right) + \frac{3}{4}(1-\psi-k)\overline{X}_x\varkappa, \\ \frac{\delta M_2}{D} &= -(1-\psi)\left(\varkappa_2 + \frac{1}{2}\varkappa_1\right) + \frac{3}{4}(1-\psi-k)\overline{Y}_y\varkappa, \\ \frac{\delta H}{D} &= -\frac{1}{2}(1-\psi)\varkappa_3 + \frac{3}{4}(1-\psi-k)\overline{X}_y\varkappa, \end{aligned} \tag{2.4}$$

where it is designated that:

$$k = \frac{4(1-\lambda)}{[1+\sqrt{1-\lambda}]^2} = \frac{4\frac{d\sigma_i}{de_i}}{\left(\sqrt{E}+\sqrt{\frac{d\sigma_i}{de_i}}\right)^2}, \tag{2.5}$$

$$\begin{aligned} \psi &= \frac{\omega}{2}\left[1-\bar{z}_0^3 + \frac{3}{4}\frac{\omega(1-\bar{z}_0^2)^2}{2-\omega+\omega\bar{z}_0}\right] = \\ &= \omega\left(1-\frac{1}{2}\sqrt{k}\right)\left[\left(1-\frac{1}{2}\sqrt{k}\right)^2 + \frac{3}{4}\frac{k}{1-\left(1-\frac{1}{2}\sqrt{k}\right)\omega}\right]. \end{aligned} \tag{2.6}$$

Equations (2.4) give the following expression for the tangential flexing moment,

$$\delta M_2 = -D\left[1-\psi-\frac{3}{4}(1-\psi-k)\bar{Y}_y^2\right]\varkappa_2, \tag{2.7}$$

or, since under the condition (2.3)

$$\sigma_i^2 = \frac{3}{4}Y_y^2,\quad \bar{Y}_y = -\frac{2}{\sqrt{3}},$$

then

$$\delta M_2 = -kD\varkappa_2. \tag{2.8}$$

It is interesting to note that from all possible values of $\bar{Y}_y$ under condition (2.3) the least rigidity of shell is obtained. If the loads acting on the shell do not satisfy condition (2.3), then expression of moment $\delta M_2$, determined by formula (2.7), can be assumed to be approximate. Furthermore, from the condition of equilibrium of internal moment $\delta M_2$ and moment of external pressure q in any section θ we have:

$$\delta M_2 = qRw + c = c - h\sigma_i\bar{Y}_y w. \tag{2.9}$$

Comparing this expression with (2.7), we obtain a differential equation

$$\frac{d^2w}{d\theta^2} + \left\{1 + \frac{-\bar{Y}_y\sigma_i R^2 h}{D\left[1-\psi-\frac{3}{4}(1-\psi-k)\bar{Y}_y^2\right]}\right\} w = c',$$

where c and c' are interconnected arbitrary constants. The least value of the expression, enclosed in braces and corresponding to the periodic with respect to θ change of w, will be $\frac{\pi^2}{4}$. Thus, using expression of flexibility i and selecting as the characteristic value of dimension of $l$ the length of circumference $2\pi R$ ($l = 2\pi R$), we obtain the critical value

$$i = \pi\sqrt{\frac{3E}{\sigma_i}\,\frac{4(1-\psi)-3(1-\psi-k)\bar{Y}_y^2}{-\bar{Y}_y}}. \tag{2.10}$$

In particular, under condition (2.3):

$$l = \pi \sqrt{\frac{6\sqrt{3}Ek}{\sigma_i}}. \qquad (2.11)$$

In the absence of axial force ($X_x = 0$, $Y_y = -\sigma_i$) we have:

$$l = \pi \sqrt{\frac{3E}{\sigma_i}(1 - \varphi + 3k)}. \qquad (2.12)$$

Now we will consider the axisymmetrical form of loss of stability of a cylindrical shell, compressed by axial force P and lateral pressure q.

Stresses prior to the loss of stability are expressed by formulas (2.1). From the condition of symmetry and equation of equilibrium in the direction of x axis it follows:

$$\delta S = \delta T_1 = 0, \quad \varepsilon_3 = x_3 = 0.$$

The exact solution of the problem posed will be obtained for that case, when the axial compressing stress is twice as large as the tangential stress:

$$X_x = 2Y_y, \quad P = 4\pi R^2 q. \qquad (2.13)$$

In this case $\bar{S}_y = 0$, and therefore, from formula (1.26) we have $\varphi = 0$, i.e., the relative thickness of plastic layer $\zeta$ is constant; formulas (1.27), (1.28) and (1.25) give:

$$\bar{z}_0 = 1 - 2\zeta = -1 + \sqrt{k}. \qquad (2.14)$$

From (1.21) we will find $\delta T_2$ and $\varepsilon_1 + \frac{1}{2}\varepsilon_2$:

$$\frac{4\delta T_2}{Eh} = 2(2 - \omega + \omega\bar{z}_0)\varepsilon_2 + \frac{\omega h}{2}(1 - \bar{z}_0^2)x_2 + \frac{(\lambda - \omega)h}{3}\bar{S}_y(1 - \bar{z}_0)^2 x,$$

$$-2(2 - \omega + \omega\bar{z}_0)\left(\varepsilon_1 + \frac{1}{2}\varepsilon_2\right) = \frac{h}{2}\left[\omega(1 - \bar{z}_0^2)\left(x_1 + \frac{1}{2}x_2\right) + \right.$$

$$\left. + \frac{3}{4}(\lambda - \omega)\bar{X}_x(1 - \bar{z}_0)^2 x. \right. \qquad (2.15)$$

These formulas are somewhat simplified when $\bar{S}_y = 0$; when $\bar{z}_0 = -1 + \sqrt{k}$ they can be considered as approximate for arbitrary values of $\bar{S}_y$ also. From the first equations of group (1.22) we have the expression for the bending moment $\delta M_1$:

$$\frac{4\delta M_1}{D} = -2(2-\omega+\omega\bar{z}_0^3)\left(\varkappa_1+\frac{1}{2}\varkappa_2\right)+\frac{3}{4}(\lambda-\omega)(1-\bar{z}_0)^2 \times$$
$$\times(2+\bar{z}_0)\bar{X}_x\varkappa-\frac{6\omega}{h}(1-\bar{z}_0^3)\left(\varepsilon_1+\frac{1}{2}\varepsilon_2\right). \tag{2.16}$$

We will designate the sag of the shell with w(x); then distortions $\varkappa_1$, $\varkappa_2$ and tangential deformation $\varepsilon_2$ will be expressed as:

$$\varkappa_1=\frac{d^2w}{dx^2},\quad \varkappa_2=\frac{w}{R^2},\quad \varepsilon_2=-\frac{w}{r},$$
$$\varkappa=\bar{X}_x\varkappa_1+\bar{Y}_y\varkappa_2=\bar{X}_x\left(\varkappa_1+\frac{1}{2}\varkappa_2\right)+\bar{S}_y\varkappa_2. \tag{2.17}$$

Excluding $\varepsilon_1 + \frac{1}{2}\varepsilon_2$ from (2.16), we find the following expressions for $\delta M_1$ and $\delta T_2$ through w;

$$\frac{\delta M_1}{D}=-(1-\psi-\chi\bar{X}_x^2)\left(\frac{d^2w}{dx^2}+\frac{w}{2R^2}\right)+\chi\bar{X}_x\bar{S}_y\frac{w}{R^2},$$
$$\frac{\delta T_2}{Eh}=-\left(1-\omega+\frac{1}{2}\omega\sqrt{k}\right)\frac{w}{R}+\frac{1}{8}h\omega\sqrt{k}(2-\sqrt{k})\frac{w}{R^2}+$$
$$+\frac{1}{8}h(\lambda-\omega)(2-\sqrt{k})^2\bar{S}_y\left(\bar{X}_x\frac{d^2w}{dx^2}+\bar{Y}_y\frac{w}{R^2}\right), \tag{2.18}$$

where function $\psi$ is expressed through (2.6), and $\chi$ has the value:

$$\chi=\frac{3}{16}(\lambda-\omega)(2-\sqrt{k})^2\left[1+\sqrt{k}+\frac{3k\omega}{4\left(1-\omega+\frac{1}{2}\omega\sqrt{k}\right)}\right]. \tag{2.19}$$

Formulas (2.18) are significantly simplified under condition (2.13), when,

$$\bar{S}_y=0,\quad \bar{Y}_y=\frac{1}{2};\ \bar{X}_x=-\frac{1}{\sqrt{3}};$$
$$\delta M_1=-kD\left(\frac{d^2w}{dx^2}+\frac{w}{2R^2}\right),\quad \delta T_2=-Eh\left(1-\omega+\right.$$
$$\left.+\frac{1}{2}\omega\sqrt{k}\right)\frac{w}{R}. \tag{2.20}$$

For solving the problem on stability it remains to write the differential equation of equilibrium,

$$\frac{d^2\delta M_1}{dx^2}+T_1\frac{d^2w}{dx^2}+\frac{\delta T_2}{R}=0. \qquad (2.21)$$

It is easily integrated in the general case, when $\delta M_1$ and $\delta T_2$ are determined by formulas (2.18), and especially in the case when $\overline{S}_y = 0$. Introducing (2.20) here, we will obtain:

$$\frac{d^4w}{dx^4}+\frac{2\sigma_i i^2}{kER^2\sqrt{3}}\frac{d^2w}{dx^2}+\frac{\left(1-\omega+\frac{1}{2}\omega\sqrt{k}\right)i^2}{R^4k}w=0, \qquad (2.22)$$

where with i we designate flexibility*

$$i = \frac{3R}{h}.$$

If the length of shell is great in comparison with the radius and the ends are freely supported, then sag w can be assumed to have the form

$$w = C \sin \alpha x,$$

while the least value of the critical force is obtained from condition

$$i=\frac{E}{\sigma_i}\sqrt{3k\left(1-\omega+\frac{1}{2}\omega\sqrt{k}\right)}. \qquad (2.23)$$

The investigation of other cases of stability is based on application either of equation (2.22), or (2.21) with values $\delta M_1$, $\delta M_2$ according to (2.18); it is entirely analogous to investigation of corresponding elastic problems, inasmuch as differential equation (2.21) is linear and contains only even derivative of w.

The problem of stability of the circular cylindrical shell was very comprehensively investigated by V. I. Korolev; he inspected the

*Here terms of the order of $\frac{h}{R}$ in comparison with 1 are rejected.

stability of the shell under axial compression, during the simultaneous action of internal pressure and axial compression, during axial compression and presence of preliminary internal pressure, etc.

Recently A. A. Il'yushin [71] proposed a new formulation of the problem of stability of thin-walled structures containing rod elements, for the case, when it is in the elastoplastic state.

## § 3. Approximate Method of Investigation of the Stability of Shells Taking into Account the Physical and Geometric Nonlinearity

A study of the stability of the shell "in the broad view" considers only geometric nonlinearity which signifies, as was noted above, the retention of quadratic terms in the series for expression of deformations through transpositions, for instance:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - k_1 w.$$

Here, independently of the value of load it was always assumed that the material, in the process of deformation remains elastic, i.e., the relationship between stresses and deformations is linear.

However, depending upon the geometry of the shell and properties of its material, with certain values of loads deformations can appear, which cannot be described by linear relationships $\sigma$ - $\varepsilon$, and then it is necessary in calculations to consider other (in general, nonlinear) relationship between $\sigma$ and $\varepsilon$, which takes into account the change of physical properties of the material in the process of loading. Attempts were made to account for this nonlinear relationship between $\sigma$ and $\varepsilon$ by means of introduction in the calculation of Prandtl's diagram or diagram with linear strengthening.

Of some interest to us, from the point of view of necessity of obtaining of a more complete concept of the work of shells, is the

allowance, in the theory of their calculation in corresponding stages of the stressed state and deformation for both forms of nonlinearity — the geometric and the physical. The complexity of the problem is evident even from the fact that from the moment of appearance of plastic deformations it is already necessary to take into consideration the discrepancy between the laws of loading and unloading. Such an allowance leads to very cumbersome calculations.

The problem is somewhat simplified, if one were to construct a theory based on the hypothesis of nonlinear-elastic material, assuming the coincidence of laws of loading and unloading.

With such formulation of the problem the elastoplastic properties of the material are not considered. Nonetheless, results of the solution can be applied to a broad class of materials (for instance, alloys, plastics, steel in the reinforcing zone in the case of active deformation, and others).

Let us give the results of research by I. A. Lukash [72], which assumes for calculation of sloping shells that Kirchhoffs'-Love's, hypotheses are just, and assumes the material to be nonlinearly-elastic, and that the following laws:

$$\sigma = \sigma(\varepsilon) \text{ and } \sigma_i = \sigma_i(e_i), \tag{3.1}$$

coincide, which takes place, for incompressible material ($\nu = 0.5$). Relationship (3.1) may be written in a sufficiently general form:

$$\sigma = \sum_{i=1}^{i=n} A_i \varepsilon_i^{k_i}, \tag{3.1}$$

where $A_i$ and $k_i$ are certain constants. In examining particular cases of this relationship:

a) $\sigma = A\varepsilon^k$ (3.2)

A and k are determined experimentally from the examination of diagram of stretching (compression),

b) $\sigma = c\varepsilon(1 - \alpha\varepsilon)$ (3.3)

constants c, m and $\alpha$ can be found according to the conventional diagram of stretching.

Geometric nonlinearity is taken into consideration, as in § 2, by introduction into the examination of quadratic terms in expressions of deformations through displacements of the middle surface:

$$\begin{aligned} e_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - k_1 w, \\ e_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - k_2 w, \\ e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \end{aligned} \qquad (3.4)$$

On the basis of Kirchhoff-Love's hypothesis deformation of an element of a shell at distance z from the middle surface will be:

$$\begin{aligned} \varepsilon_x &= e_x + z\varkappa_x \quad \left(\varkappa_x = -\frac{\partial^2 w}{\partial x^2}\right), \\ \varepsilon_y &= e_y + z\varkappa_y \quad \left(\varkappa_y = -\frac{\partial^2 w}{\partial y^2}\right), \\ \varepsilon_{xy} &= e_{xy} + 2z\varkappa_{xy} \quad \left(\varkappa_{xy} = -\frac{\partial^2 w}{\partial x\,\partial y}\right). \end{aligned} \qquad (3.5)$$

Introducing (3.4) and (3.5) in the expression for intensity of deformations (when $\nu = 0.5$):

$$e_i = \frac{2}{\sqrt{3}}\sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_x\varepsilon_y + \frac{1}{4}\varepsilon_{xy}^2} \qquad (3.6)$$

we will obtain:

$$e_i = \frac{2}{\sqrt{3}}\sqrt{b_1 + b_2' z + b_3' z^2}, \qquad (3.7)$$

where

$$\begin{aligned} b_1 &= e_x^2 + e_y^2 + e_x e_y + \frac{1}{4} e_{xy}^2, \\ b_2' &= 2e_x\varkappa_x + 2e_y\varkappa_y + e_x\varkappa_x + e_y\varkappa_x + e_{xy}\varkappa_{xy}, \\ b_3' &= \varkappa_x^2 + \varkappa_y^2 + \varkappa_x\varkappa_y + \varkappa_{xy}^2. \end{aligned} \qquad (3.8)$$

Introducing dimensionless variable

$$t = \frac{2z}{h}, \quad (-1 < t < +1) \tag{3.9}$$

we have:

$$\varepsilon_x = e_x + \varkappa_1 t; \quad \varepsilon_y = e_y + \varkappa_2 t; \quad \varepsilon_{xy} = \varkappa_{12} t. \tag{3.10}$$

where

$$\varkappa_1 = \varkappa_x \frac{h}{2}; \quad \varkappa_2 = \varkappa_y \frac{h}{2}; \quad \varkappa_{12} = \varkappa_{xy} h. \tag{3.11}$$

Further

$$e_i = \frac{2}{\sqrt{3}} \sqrt{b_1 + b_2 t + b_3 t^2}. \tag{3.12}$$

where

$$\begin{gathered} b_1 = e_x^2 + e_y^2 + e_x e_y + \frac{1}{4} e_{xy}^2, \\ b_2 = 2e_x \varkappa_1 + 2e_y \varkappa_2 + e_x \varkappa_2 + e_y \varkappa_1 + \frac{1}{2} e_{xy} \varkappa_{12}, \\ b_3 = \varkappa_1^2 + \varkappa_2^2 + \varkappa_1 \varkappa_2 + \frac{\varkappa_{12}^2}{4}. \end{gathered} \tag{3.13}$$

Deformation energy of nonlinearly-elastic and elastoplastic body according to [2] is given by formula

$$V = \iiint \left[ \int_0^{e_i} \sigma_i \, de_i + \frac{k\theta^2}{2} \right] dx\, dy\, dz, \tag{3.14}$$

where

$$k = \frac{e_1 + e_2 + e_3}{2}, \quad \theta = \varepsilon_x + \varepsilon_y + \varepsilon_z.$$

For an incompressible body the volume deformation $\theta = 0$ and (3.14) will be written as follows

$$V = \iiint \left[ \int_0^{e_i} \sigma_i \, de_i \right] dx\, dy\, dz \tag{3.15}$$

or taking into account $\sigma_i = \sigma_i(e_i)$ –

$$V = \iiint \left[ \int_0^{e_i} f(e_i)\, de_i \right] dx\, dy\, dz. \tag{3.16}$$

The work of external forces W is determined by the formula

$$W = \iint (q_x u + q_y v + q_z w)\, dx\, dy \tag{3.17}$$

($q_x$, $q_y$, $q_z$ are components of loads in the directions x, y, z).

Equating the variation of external work to the variation of the work of internal forces, we will obtain the relationship

$$\delta W = \delta V, \tag{3.18}$$

which interconnects four functions.

1. Function $F_0$ of the middle surface of shell with an initial load (this function is included in (3.18) through curvatures of the sloping shell):

$$k_x = \frac{\partial^2 F_0}{\partial x^2}, \quad k_y = \frac{\partial^2 F_0}{\partial y^2}, \quad k_{xy} = \frac{\partial^2 F_0}{\partial x \, \partial y},$$

which are included in (3.4) and (3.6);

2. Function $F_q$ of the middle surface of the shell with load q. In equation (3.18) this function is included through transpositions of u, v, w;

3. Function of load q(x, y, z);

4. Function $\sigma(e_i)$, which describes the physical properties of the material.

Let us assume that we are given the function $\sigma_i = \sigma(e_i)$, the shape of shell prior to loading $F_0 = F_0(x, y)$ and the function of load q(x, y); it is required to determine the shape of the surface of the shell after loading. Let us examine an approximate solution of this problem, assuming the exponential dependence of stresses on deformations. Substituting (3.2) in (3.15), we will obtain:

$$V = \frac{A}{k+1} \iiint e_i^{k+1} \, dx \, dy \, dz. \tag{3.19}$$

Here the triple integral extends onto the entire volume of the shell. When k = 1 and A = E we will obtain the deformation energy of a linearly-elastic body:

$$V = \frac{E}{2}\iiint e_i^2\,dx\,dy\,dz. \tag{3.20}$$

When $k = 0$, $A = \sigma_s$ we will have the deformation energy of a rigidly plastic body

$$V = \sigma_s \iiint e_i\,dx\,dy\,dz. \tag{3.21}$$

Let us introduce in (3.19) the value of the intensity of deformation from (3.12) and, taking into account (3.9), we will obtain the expression for deformation energy:

$$V = \frac{2^k A h}{(k+1)3^{\frac{k+1}{2}}}\int_{-a}^{+a}\int_{-b}^{+b}\int_{-1}^{+1}(b_1 + b_2 t + b_3 t^2)^{\frac{k+1}{2}}\,dx\,dy\,dt. \tag{3.22}$$

Introducing designations

$$e^2 = b_1 + b_2 t + b_3 t^2, \tag{3.23}$$

we will write deformation energy in the following form

$$V = \frac{2^k A h}{(k+1)3^{\frac{k+1}{2}}}\int_{-a}^{+a}\int_{-b}^{+b}\int_{-1}^{+1} e^{k+1}\,dx\,dy\,dt. \tag{3.24}$$

Integration with respect to t can be performed by Simpson's formula:

$$\int_{-1}^{+1} e^{k+1}\,dt = \frac{3}{3}\left[\frac{e_b^{k+1} + e_n^{k+1}}{2} + 2e_0^{k+1}\right] = \frac{1}{3}(e_b^{k+1} + e_n^{k+1} + 4e_0^{k+1}), \tag{3.25}$$

where values

$$e_b = \sqrt{b_1 + b_2 + b_3},\quad e_0 = \sqrt{b_1},\quad e_n = \sqrt{b_1 - b_2 + b_3} \tag{3.26}$$

are obtained from (3.23) after substituting in it respectively

$t = +1$, $t = 0$, $t = -1$.

Finally for the deformation energy we will obtain

$$V = \frac{2^k}{3^{\frac{k+3}{2}}}\cdot\frac{Ah}{k+1}\int_{-a}^{+a}\int_{-b}^{+b}(e_b^{k+1} + e_n^{k+1} + 4e_0^{k+1})\,dx\,dy; \tag{3.27}$$

here the double integral is taken with respect to the entire surface S of the shell. Executing the integration, we will obtain

$$I=\iint_S F\,dx\,dy=\frac{8}{3}F_0+\frac{1}{3}(F_1+F_2+F_3+F_4), \tag{3.27'}$$

where $F_i$ are values of integrand in angular points of the square with side 2. For the shell, square in plan with side 2a, which is under the action of an evenly distributed load with symmetric boundary conditions, we will obtain

$$I=\frac{a^2}{3}(8F_0+F_1+F_2+F_3+F_4). \tag{3.28}$$

If, for instance,

$$F=\cos\frac{\pi y}{2a}\cos\frac{\pi x}{2a},$$

then

$$I=\int_{-a}^{+a}\int_{-a}^{+a}\cos\frac{\pi y}{2a}\cos\frac{\pi x}{2a}\,dx\,dy=\frac{16a^2}{\pi^2} \tag{3.29}$$

will be the exact value of the integral. According to formula (3.28) we will obtain

$$F_0=\frac{1}{2};\quad F_1=F_2=F_4=0;\quad F_3=1;$$

$$I=\frac{a^2}{3}\left(8\,\frac{1}{2}+1\right)=\frac{5}{3}a^2,$$

which as compared to (3.29) gives an error of 3%. Taking into consideration (3.27), the expression for energy can be given in the following form

$$V=\frac{2^k}{(k+1)3^{\frac{k+5}{2}}}Aabh(8F_0+F_1+F_2+F_3'+F_4). \tag{3.30}$$

Here

$$F_i=e_b^{k+1}+e_H^{k+1}+4e_0^{k+1},\quad i=0,1,2,3,4, \tag{3.31}$$

and values $e_b$, $e_H$, $e_0$ are determined by formulas (3.26). If only

one transverse load is in effect, then the work of external forces will be equal to

$$W = \int_{-a}^{+a}\int_{-b}^{+b} qw\,dx\,dy. \tag{3.32}$$

Total energy of system

$$U = U(u, v, w) = V - W. \tag{3.33}$$

Let us present displacements in the form of series:

$$u = \sum c_i\bar{u}_i, \quad v = \sum c'_i\bar{v}_i, \quad w = \sum w_i\bar{w}_i, \tag{3.34}$$

where $\bar{u}_i$, $\bar{v}_i$, and $\bar{w}_i$ are the prescribed functions of displacements, satisfying boundary conditions, and $c_i$, $c'_i$ and $\bar{W}_i$ are the coefficients sought. Introducing (3.34) into (3.33) and setting up the conditions of extremum of energy U:

$$\frac{\partial U}{\partial c_i} = 0; \quad \frac{\partial U}{\partial c'_i} = 0, \quad \frac{\partial U}{\partial W_i} = 0, \tag{3.35}$$

we will obtain systems of algebraic equations for determination of coefficients $c_i$, $c'_i$ and $w_i$. This system of nonlinear equations with fractional indices may be solved by approximation or graphic methods. After determination of coefficients $c_i$, $c'_i$ and $w_i$ it is not difficult to find from relationships (3.5) and (3.4) functions of displacements and deformations.

Upon the solution of this problem it is very important to select suitable functions $\bar{u}_i$, $\bar{v}_i$ and $\bar{w}_i$ in such a manner that they would satisfy kinematic boundary conditions and describe as well as possible the deformed surface of shell. It is useful to present functions $\bar{u}_i$, $\bar{v}_i$, $\bar{w}_i$ in the form of the product of two functions, each of which depends only on one coordinate:

$$\bar{u}_i = \bar{u}_i(x)\cdot\bar{u}_i(y); \quad \bar{v}_i = \bar{v}_i(x)\cdot\bar{v}_i(y); \quad \bar{w}_i = \bar{w}_i(x)\cdot\bar{w}(y). \tag{3.36}$$

For the latter we can select, for instance, beam fundamental functions, corresponding to the boundary conditions. The method presented is rather labor-consuming for the calculation of shells taking into account the physical and geometric nonlinearity.

Let us examine a shell with displacing edges. In this case it is possible to construct an approximate solution, applying a simplified formula for the intensity of deformations:

$$e_i = \frac{2}{\sqrt{3}}(e_x + e_y). \tag{3.37}$$

Introducing here deformation values from (3.5), we will obtain

$$e_i \approx \frac{2}{\sqrt{3}}(e_x + \varkappa_x z + e_y + \varkappa_y z) = \frac{2}{\sqrt{3}}(e + \varkappa z); \tag{3.38}$$

here

$$e = e_x + e_y; \quad \varkappa = \varkappa_x + \varkappa_y. \tag{3.39}$$

Introducing (3.38) in (3.19), we will obtain the following expression for the work of internal forces:

$$V = \frac{A 2^{k+1}}{(k+1)3^{\frac{k+1}{2}}} \iiint\limits_{-\frac{h}{2}}^{+\frac{h}{2}} (e + \varkappa z)^{k+1}\, dx\, dy\, dz, \tag{3.40}$$

or after integration with respect to z:

$$V = \frac{2^{k+1}}{(k+1)(k+2)3^{\frac{k+1}{2}}} \iint \frac{\left(e + \varkappa\frac{h}{2}\right)^{k+2} - \left(e - \varkappa\frac{h}{2}\right)^{k+2}}{\varkappa}\, dx\, dy. \tag{3.41}$$

where integration extends over the entire area of the supporting plan of the shell. Let us find the relationship between the load and sag. Being limited in (3.27) by one zero point, we will obtain

$$V = \frac{2^{k+4}}{(k+1)(k+2)3^{\frac{k+1}{2}}} \cdot \frac{\left(e_0 + \varkappa\frac{h}{2}\right)^{k+2} - \left(e_0 - \varkappa_0\frac{h}{2}\right)^{k+2}}{\varkappa_0}. \tag{3.42}$$

Here $e_0$ and $\varkappa_0$ are values of magnitudes (3.39) at point $\left(\frac{a}{2}, \frac{b}{2}\right)$. Further, taking into consideration (3.4), we will have:

$$e_0 = (c + c_1) a_0 - (k_x + k_y) w_0 \bar{w}_0 + \frac{w_0^2}{2} e. \tag{3.43}$$

where $a_0$, $w_0$, $b_0$ and e are unknown coefficients.

Here it is assumed that

$$u = c\bar{u}, \quad v = c_1\bar{v}, \quad w = w_0 \cdot \bar{w}. \tag{3.44}$$

Further, let us assume that the relationship between coefficients c, $c_1$ and $w_0$ is the same, as in the elastic theory. Let us assume that:

$$\begin{aligned} c &= w_0 k_x a_1 - \frac{w_0^2}{2} a_2, \\ c_1 &= w_0 k_y b_1 - \frac{w_0^2}{2} b_2, \end{aligned} \tag{3.45}$$

where $a_1$, $a_2$, $b_1$, $b_2$ are coefficients to be determined. From (3.45) we have,

$$c + c_1 = w_0 (k_x a_1 + k_y b_1) - \frac{w_0^2}{2} (a^2 + b^2). \tag{3.46}$$

Introducing (3.46) into (3.44), we will obtain:

$$e_0 = w_0 (k_x + k_y) S + \frac{w_0^2}{2} m; \tag{3.47}$$

here

$$S = \frac{a_0 (k_x a_1 + k_y b_1)}{k_x + k_y} - \bar{w}_0; \quad b_0 - ( \ . + b_2) a_0 = m; \tag{3.48}$$

taking into account (3.17) and (3.29) for the work of external forces and taking into consideration the relationships given, we will obtain the following expression for the total energy of the system:

$$U = \frac{2^{k+4} A a b}{(k+1)(k+2) 3^{\frac{k+3}{2}}} \cdot \frac{\left(e_0 + \varkappa_0 \frac{h}{2}\right)^{k+2} - \left(e_0 + \varkappa_0 \frac{h}{2}\right)^{k+2}}{\varkappa_0} - $$

$$- q w_0 \frac{16ab}{\pi^3}, \tag{3.49}$$

taking the derivative with respect to $w_0$, we will obtain the relationship between load and sag, which after substitution of values (3.44) and (3.47) will assume the following form (the formula is written for a cylindrical shell square in plan, when $k_y = 0$, $b = a$):

$$q = \frac{2^k a^2 A}{(k+1)(k+2)3^{\frac{k+3}{2}}} \cdot \frac{B^{k+1}[cw_0(k+2) - B] - \bar{B}^{k+1}[\bar{c}w_0(k+2) - \bar{B}]}{w_0^2 c_0}, \tag{3.50}$$

where

$$\left.\begin{aligned} B &= w_0 k_x s + \frac{w_0^2}{2} m + \frac{h}{2} c_0 w_0. \\ \bar{B} &= w_0 k_x s + \frac{w_0^2}{2} m - \frac{h}{2} c_0 w_0. \\ c &= w_0 b_0 - \bar{w}_0 k_x + \frac{h}{2} c_0. \\ \bar{c} &= w_0 b_0 - \bar{w}_0 k_x - \frac{h}{2} c_0. \end{aligned}\right\} \tag{3.51}$$

values s, m, b, c and $\bar{w}_0$ still remain unknown and will have to be determined.

Let us introduce dimensionless parameters:

$$\left.\begin{aligned} & p = \frac{q(2a)^4}{Eh^4}; \quad \frac{w_0}{h} = \xi; \quad \frac{f_0}{h} = \xi_0; \\ & k_x = \frac{1}{R} \approx \frac{2f_0}{a^2} = \frac{2\xi_0 h}{a^2}. \end{aligned}\right\} \tag{3.52}$$

In these formulas $f_0$ is the initial rise, and a is one half of the side of the supporting square plan of the shell. Let us assume, in accordance with their dimensions, that:

$$m = \frac{\bar{m}}{a^2}; \quad b_0 = \frac{\bar{b}_0}{a^2}; \quad c_0 = \frac{\bar{c}_0}{a^2}. \tag{3.53}$$

Substituting (3.53) and (3.52) in (3.50), we will obtain:

$$\left.\begin{aligned} B &= \frac{h^2}{a^2}\left[2s\xi_0 + \frac{\bar{m}}{2}\xi + \frac{\bar{c}_0}{2}\right]\xi = \frac{h^2}{a^2} b^* \xi; \\ \bar{B} &= \frac{h^2}{a^2}\left[2s\xi_0 + \frac{\bar{m}}{2}\xi - \frac{\bar{c}_0}{2}\right]\xi = \frac{h^2}{a^2} \bar{b}^* \xi; \\ c &= \frac{h}{a^2}\left(\bar{b}_0\xi - 2\bar{w}_0\xi_0 + \frac{\bar{c}_0}{2}\right) = \frac{h}{a^2} c^*; \\ \bar{c} &= \frac{h}{a^2}\left(\bar{b}_0\xi - 2\bar{w}_0\xi_0 - \frac{\bar{c}_0}{2}\right) = \frac{h}{a^2} \bar{c}^*. \end{aligned}\right\} \tag{3.54}$$

After substitution of (3.52) and (3.54) in (3.50) we will have,

$$p = D_k\{(b^*)^{k+1}[c^*(k+2) - b^*] - (\bar{b}^*)^{k+1}[\bar{c}^*(k+2) - \bar{b}^*]\}\cdot\xi^k, \qquad (3.55)$$

where

$$\left.\begin{aligned} b^* &= 2s\xi_0 + \frac{\bar{m}}{2}\xi + \frac{\bar{c}_0}{2}; \\ \bar{b}^* &= 2s\xi_0 + \frac{\bar{m}}{2}\xi - \frac{\bar{c}_0}{2}; \\ c^* &= \bar{b}_0\xi - 2\bar{w}_0\xi_0 - \frac{\bar{c}_0}{2}; \\ \bar{c}^* &= \bar{b}_0\xi - 2\bar{w}_0\xi_0 - \frac{\bar{c}_0}{2}; \end{aligned}\right| \qquad (3.56)$$

$$D_k = \frac{A}{E}\,\frac{\pi^4\cdot 2^{k+4}}{(k+1)(k+2)3^{\frac{k+3}{2}}\bar{c}_0}\left(\frac{a}{h}\right)^{2(1-k)}. \qquad (3.57)$$

Formula (3.55) expresses the general relationship between dimensionless load p and sag $\xi$ with an accuracy up to the so far unknown coefficients $\bar{m}$, $\bar{b}_0$, $\bar{s}$, $\bar{w}_0$, $\bar{c}_0$. These coefficients will be found from the following two conditions.

1. When $k = 0$, $A = \sigma_s$ and $\xi_0 = 0$ formula (3.55) should yield the solution for the rigid plastic plate:

$$q = \frac{\pi^4 + \sigma_s w_0 h}{16\sqrt{3}a^4}. \qquad (3.58)$$

2. When $k = 0$ and $A = E$ formula (3.55) should yield the solution for the elastic shell:

$$p = \alpha_1\xi^3 + \alpha_2\xi^2\xi_0 + \alpha_3\xi\xi_0^2 + \alpha_4\xi. \qquad (3.59)$$

Setting up from formula (3.55) these conditions (1) and (2) and equating the corresponding coefficients, we will obtain a system of five equations for determination of five unknown coefficients. Solving this system, we will find the following values of coefficients $\bar{m}$, $\bar{b}_0$, $\bar{s}$, $\bar{w}_0$, $\bar{c}_0$:

$$\left.\begin{aligned} &\bar{b}_0 = \frac{3\pi^2}{16}; \quad \bar{m} = \frac{3\alpha_1}{\pi^4}; \quad s = \frac{3\alpha_3}{8\pi^4}\beta_{1,2}; \\ &\bar{w}_0 = -\frac{3\pi^2}{16}\cdot\frac{\alpha_3}{\alpha_2\beta_{1,2}}; \quad \bar{c}_0 = \frac{3\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{\alpha_4}}{\pi}. \end{aligned}\right\} \tag{3.60}$$

In these formulas

$$\beta_{1,2} = 1 \pm \sqrt{1 - \frac{4\alpha_1\alpha_3}{\alpha_2^2}}. \tag{3.61}$$

Since coefficient $\beta_{1,2}$ has two values, then there will be two relationships (3.55) also. Of these, the one which will yield the lowest load value for the same sag, will be used for the calculation.

As a numerical example let us find a relationship (3.55) for the cylindrical shell, square in plan, with the following dimensions:

$$a = 20\ \text{см}; \quad h = 0{,}7\ \text{см}; \quad f_0 = 1{,}6\ \text{см};$$

$$\xi_0 = \frac{1{,}6}{0{,}7} = 2{,}29; \quad \xi_0^2 = 5{,}24.$$

Physical characteristics of the material, determined by the extension diagram are:

$$E = 1{,}83\cdot 10^6\ \text{kg/cm}^2; \quad A = 4{,}258\cdot 10^3\ \text{kg/cm}^2; \quad k = 0{,}137.$$

For a shell hinge-supported on the edges of coefficients $\alpha_i$ for the elastic problem will be:

$$\alpha_1 = 8{,}63; \quad \alpha_2 = -19{,}62; \quad \alpha_3 = 9{,}92;$$

$$\alpha_4 = 22{,}12\,\frac{0{,}91}{1-\mu^2} \quad (\mu^2 = 0{,}5).$$

Using formula (3.61) we will calculate coefficient

$$\beta_{1,2} = 1 \pm \sqrt{1 - \frac{4\cdot 8{,}63\cdot 9{,}92}{19{,}62^2}} = 1 \pm 0{,}332;$$

$$\beta_1 = 1{,}332; \quad \beta_2 = 0{,}668.$$

Using formulas (3.60) we calculate coefficients $\bar{b}_0$, $\bar{m}$, $s$, $\bar{w}_0$, $\bar{c}_0$:

$$\bar{b}_0 = 18{,}5; \quad \bar{m} = 0{,}266; \quad s_1 = -0{,}1066;$$

$$\bar{w}_{01} = +0{,}02; \quad \bar{c}_0 = +3{,}0295; \quad s_2 = -0{,}0504; \quad \bar{w}_{02} = +1{,}401.$$

Substituting these values first in (3.56), and then in (3.55), we will obtain

$$\left.\begin{array}{c} \text{for } \beta_1 = 1,332 \\ p_1 = D_k [(0,133\xi + 1,054)^{1,137}(3,8221\xi - 4,691) - \\ - (0,133\xi - 1,976)^{1,137}(3,822\xi - 8,135)]\xi^{0,137}; \\ \text{for } \beta_2 = 0,668 \\ p_2 = D_k [(0,133\xi + 1,284)^{1,137}(3,822\xi - 11,762) - \\ - (0,133\xi - 1,746)^{1,137}(3,822\xi - 15,204)]\xi^{0,137}. \end{array}\right| \quad (3.62)$$

Value $D_{0,137}$ will be found by the formula (3.57):

$$D_{0,137} = \frac{4,258 \cdot 10^{3}}{1.83 \cdot 10^{6}} \cdot \frac{2^{4,137}\pi^2}{1,137 \cdot 2,137 \cdot 3^{\frac{3,137}{2}} \cdot 3,0295}\left(\frac{20}{0,7}\right)^{2(1-0,137)} = 3,1871.$$

Graphs of relationships (3.62) are plotted in Fig. 60 with solid lines $\beta_1$ and $\beta_2$.

The calculated shell was tested in an Institute of Mechanics, Academy of Sciences USSR for an evenly-distributed load. Sags in the center of shell were measured during the tests. Experimental p - $\xi$ relationships for two samples were also plotted in Fig. 60 with dotted lines curve I, calculated by the formula (3.59) was plotted above them.

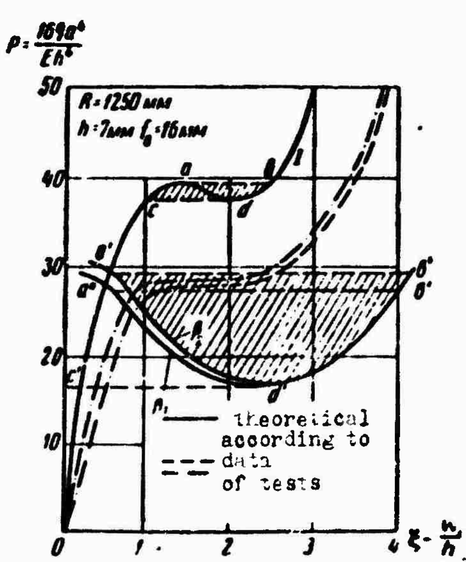

Fig. 60.

As can be seen from the picture, curve I, calculated according to the linear-elastic theory, but with physical linearity, lies higher than the experimental curves. This is explained by the fact that in the given shell large sags are accompanied by large deformations, a significant part of which lies in zone of hardening.

Relationships (3.62) yield curves $\beta_1$ and $\beta_2$ which, when

deformations are small, are located higher than the experimental curve and even higher than curve 1, calculated according to the linear-elastic theory. This is explained by the fact that when deformations are small formula (3.2) yields larger values of elastic modulus than those, which are obtained from Hooke's law. When deformations are large this curve is located lower than the experimental curve.

Curves 1 and $\beta_1$, $\beta_2$ intersect in points $\alpha'$ and $\alpha''$. This part of curves $\beta_1$ and $\beta_2$, which is located to the left of points $\alpha'$ and $\alpha''$, must be rejected, since in this zone the material follows Hook's law; the behavior of the shell in this sector is described by curve 1. To the right of point $\alpha'$ the behavior of the shell will be described by curves $\beta_1$ and $\beta_2$. Thus, theoretically the behavior of the shell in the range of considered deformations examined is described by two curves: curve 1 and either one of curve $\beta_1$ or $\beta_2$.

Crosshatched zones abcd for curve 1 and a'b'd'c' or a"b'd'c' for curves $\beta_1$ and $\beta_2$ in Fig. 60 are zones of instability. Within the limits of these zones snapping of the shell should occur.

Physically linear and geometrically nonlinear calculation (curve 1) yields a narrow zone of instability. According to this example we will find that the critical force, with which snapping of the shell occurs, should lie between p = 37.8 and p = 39.8. Calculation taking into consideration both forms of nonlinearity yields a wider zone of instability; in this case the critical force should lie between p = 16.5 and p = 29.5 on curve $\beta_2$ and p = 28 on curve $\beta_1$. The experimental value of the critical force for both shells tested, corresponding to the horizontal sections of dotted lines, proved to be equal on the average to $p \approx 28.5$. From the drawing it is clear that this value lies within the limits of the

zone of instability, given not by curve 1, but by curve $\beta_2$, and differs from the upper value of the theoretical critical force by 3.5%, and from the lower value, by 72.5%.

Curve $\beta_1$ gives us the value of the upper critical force which is 1.5% less than the experimental value. The average theoretical value of the upper critical force, calculated according to curves $\beta_1$ and $\beta_2$, equal to $p = 28.75$, is almost the same as the experimental value $p = 28.5$. It must be noted that according to the conditions of the experiment (through preparation of the structure and static load) it is more probable that the shell will change to a new state of equilibrium after achievement of the upper and not the lower critical value.

This comparison with the results of experiments confirms the necessity of taking into account the physical nonlinearity, since this accounting yields a better congruence with the experiment than calculation according to the elastic nonlinear theory.

Part of curves $\beta_1$ and $\beta_2$, located to the right of point d', lies nevertheless significantly lower than the experimental curve, although it coincides with it in its character. These divergences are explained by the fact that in the given theory the material is considered to be nonlinearly-elastic; in reality steel in the strengthening zone is an elastoplastic material. In other words, the divergence in lengths of the snapping sections and in the value after the critical load is explained by the fact that the given theory does not take into consideration the influence of unloading.

## § 4. A. V. Pogorelov's Method for Investigation of the Supercritical State of Shells

### 1. Presentation of Elastic Buckling of a Shell by Means of Geometric Mirror Buckling

Let us assume that on the surface we can build a u, v coordinate network such that the vector-function r(u, v), giving the surface in these coordinates, is a regular (at least twice differentiable) function. Surfaces, for which linear elements, i.e., differential quadratic forms,

$$ds^2 = dr^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

are identical, are called isometric. This means that lengths of corresponding curves, plotted between corresponding points of the two surfaces, are identical.

Surface F will be single-valued and definite, if among the surfaces of a given class every surface, isometric with F, is equal to F.

In surface deformation its linear element $ds_t^2$ in a general instance changes and is a function of parameter t; however there exist deformations, with which changes of linear element do not occur, and consequently, there are no changes in the length of curves on the surface. Such deformations are called deflections of the surface. A surface is called rigid in a given class (for instance, the class of convex surfaces), if any bending, which does not take it from this class, is reduced to motion of the surface as a solid body. If edges of the surface are clamped or supported with hinges then there is inflexibility and a single-valued indeterminate form of surface F. We can easily prove [73] that the isometric transformation of a regular surface secured at the edge, in the class of piecewise-regular surfaces, is reduced to mirror buckling, i.e., to the reflection of its arbitrary segment in the plane, which cuts it

off (Fig. 61).

Further, let us assume that the internal metrics of the convex regular surface F changes slightly, i.e., the deformation of its linear element is small, –

$$\varepsilon(t) = \frac{ds_t - ds}{ds} \to 0 \qquad \text{when } t \to 0.$$

In this case surface F is deformed into surface $F_t$ similar to it. Whereas F consists of an internally convex region $G_1$, convex region $G_2$ adjoining the edge, and a certain ring-shaped band $G_{12}$, separating regions $G_1$ and $G_2$, located in $\delta(t)$ neighborhood of region $G_1$, then surface $F_t$ is similar either to surface F, or to the surface, obtained by mirror buckling from F.

On the basis of these geometrical premises, A. V. Pogorelov examines the approximation of elastic buckling of the shell in supercritical deformation by means of mirror buckling. Here it is assumed that the shell is convex with secured edges, its thickness either slowly changing or constant, and the middle surface is sufficiently regular.

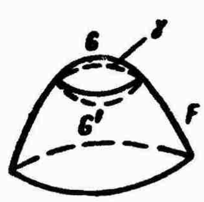


Fig. 61.

Fig. 62.

The supercritical deformation of shell is such a deformation with which the shape of the shell differs significantly from its initial shape.

Let us assume that a convex shell with thickness h is under action of certain load q, taking certain form of elastic equilibrium $F_q$. An increase of the load will result in its buckling, such a type of buckling that the middle surface can be imagined (this is actually observed) as composed of three above mentioned parts, $G_1$, $G_2$ and $G_{12}$.

Let us decrease thickness h, and together with it load q in such a manner that the order of deformation examined is retained. Since the bending rigidity of shell decreases faster than tensile rigidity, the stress in the middle surface, and, consequently, its deformation will decrease. Passing to limit ($h \to 0$), we have to conclude that the elastic buckling of a strictly convex shell $F_q$ is reduced to mirror buckling, and here the smaller the thickness of the shell, the better the approximation of mirror bucking to elastic buckling (Fig. 62). Such approximation of elastic buckling in supercritical deformations enables us to linearize the problem of determination of the elastic state of the shell outside the neighborhood of the rib of mirror buckling, inside of which the problem is essentially nonlinear.

## 2. Energy of Elastic Deformation

Let us construct the energy of elastic deformation for the above mentioned parts of shell $G_1$, $G_{12}$. Region $G_1$ is that part of shell of elastic buckling in which the shape of deformed surface is well approximated by mirror buckling. In this region the bend is significant, the normal curvature changes its sign, and the middle surface is strictly convex by assumption. As we have already shown, region $G_2$ adjoins the surface edge, is well approximated by the initial form and is significantly smaller than region $G_1$, and the bend of middle surface in region $G_2$ is small. Therefore, we can disregard the bending energy in region $G_2$ as compared to bending energy in region $G_1$. Thus, the energy of elastic deformation, connected with the bend of the shell in the buckling region, is concentrated in regions $G_1$ and $G_{12}$.

Since in mirror buckling of the shell the normal curvature changes from value k to -k, i.e., by 2k (in main senses the change of curvatures occurs on $2k_1$ and $2k_2$), the energy of the shell, corresponding to such

a bend, is calculated by the formula

$$U_{G_1} = \frac{Eh^3}{6(1-\nu^2)} \iint_{G_1} [(k_1 + k_2)^2 - 2(1-\nu) k_1 k_2] \, dG_1.$$

With a sufficiently small thickness of shell, region $G_1$ converges to region G. In this case index 1 may be omitted, and the energy expression will be written in the form

$$U_G = \frac{Eh^3}{6(1-\nu^2)} \iint_G K_\nu \, dG, \tag{4.1}$$

where

$$K_\nu = (k_1 + k_2)^2 - 2(1-\nu) k_1 k_2. \tag{4.2}$$

Region $G_{12}$ is the ring-shaped region, including the boundary of buckling. The area of this region is small as compared to the area of region $G_1$, but the energy of elastic deformation will be significant because of the large flexure of shell and stretching (compression) of the middle surface. Let us assume that $\gamma$ is the intersection of plane $\alpha$ (Fig. 63) with the middle surface of the shell. Let us weaken shell $F^*$ with a hinge along $\gamma$ and apply to each of the shell's parts the distributed moment M, straightening the rib of the shell $F^*$. In the proximity of $\gamma$ elastic states of shells $F^*$ and F will be equivalent, since in this proximity elastic deformations are caused mainly by straightening of rib $\gamma$.

Consequently, the deformation of shell $F^*$ along $\gamma$ are determined by its structure near the rib and depend essentially only on the rib curvature and the angle formed along it by tangent planes.

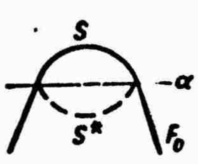


Fig. 63.

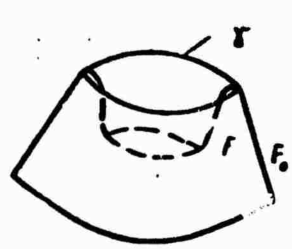

Fig. 64.

Thus, in order to find the energy of elastic deformation on the boundary of the buckling region for a sufficiently small shell thickness, we can take any other shell with the same geometric parameters of the boundary of mirror

buckling [73, 74 and 75]. Let us take the conical shell (Fig. 64) $F_0$. The energy of elastic deformation of bulged shape F of the shell near line $\gamma$ consists of the energy of stretching (compression) of the middle surface along meridians and parallels and the bending energy along meridians (for a shell of small thickness the bending energy of parallels can be disregarded).

Let us assume that $p^*$ is an arbitrary point of mirror buckling of shell $F_0$. The shape of $F^*$ will be assumed to be the initial approximation of the shape of F. Corresponding point p of shape of F is obtained in radial displacement $p^*$ on u and axial displacement on v. Hence, the equation of meridian of shell F in cylindrical coordinates r and z will be

$$r = \rho + s\cos\alpha + u, \quad z = s\sin\alpha + v,$$

where $\rho$ is the radius of circumference; $\gamma$ is the rib of shell $F^*$; s is the distance on the generator, measured from $\gamma$; and $\alpha$ is the angle, formed by generators and the plane of circle $\gamma$.

We introduce,

$$\xi = \cos\alpha, \quad \eta = \sin\alpha, \quad r_0 = \rho + s\cos\alpha.$$

The deformation of the shell along the parallel will be

$$\varepsilon_2 = \frac{2\pi(r_0 + u) - 2\pi r_0}{2\pi r_0} = \frac{u}{r_0}.$$

The deformation $\varepsilon_1$ along meridian will be

$$\varepsilon_1 = \frac{ds_1 - ds}{ds}.$$

In the region of elastic deformations $\frac{ds_1}{ds} \approx 1$ and, consequently,

$$\varepsilon_1 = \frac{ds_1^2 - ds^2}{2ds^2},$$

and since

$$ds_1^2 = dr^2 + dz^2,$$

then

$$\varepsilon_1 = \xi u' + \eta v' + \frac{1}{2}(u'^2 + v'^2).$$

The extension (compression) energy of the middle surface of the shell in supercritical deformation outside the small proximity of the zone of large flexure of the boundary buckling is equal to the energy

of elastic deformation of shell in precritical deformation, caused by the same external load [73]. Referred to one unit of surface area it is, as we well know, calculated either by the formula

$$U' = \frac{Eh}{2(1-\nu^2)}\left(\varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2\right)$$

or

$$U' = \frac{Eh}{2(1-\nu^2)}\left[(\varepsilon_1 + \nu\varepsilon_2)^2 + (1-\nu^2)\varepsilon_2^2\right]. \tag{4.3}$$

The energy of the bend along the meridian, referred to one unit of surface area, will be

$$U'' = \frac{Eh^3\varkappa^2}{24(1-\nu^2)}, \tag{4.4}$$

where $\varkappa$ is the change of meridian curvature caused by deformation. Since $\frac{ds_1}{ds} \approx 1$, then for $\varkappa$ we can use the expression

$$\varkappa = v''(\xi + u') - u''(\eta + v').$$

Thus, the energy of elastic deformation of the external half-proximity of the boundary of the region of buckling of shell F will be

$$U = \frac{Eh}{2(1-\nu^2)}\int_0^{2\pi}\int_0^{\varepsilon^*}\left\{\frac{h^2}{12}\left[v''(\xi + u') - u''(\eta + v')\right]^2 + \right.$$
$$\left. + \left(\xi u' + \eta v' + \frac{\nu u}{r_0} + \frac{u'^2 + v'^2}{2}\right)^2 + (1-\nu^2)\frac{u^2}{r_0^2}\right\} r_0\, ds\, d\theta, \tag{4.5'}$$

where $\varepsilon^*$ is the width of semiproximity, in which basically the energy of elastic deformation is concentrated, connected with straightening of the rib of shell $F^*$.

In view of smallness of $\varepsilon^*$ the expression for U will change but little, if we will replace everywhere $r_0(s)$ by $\rho$, the radius of circumference, limiting the region of buckling. The relative error here will be vanishingly minute when $h \to 0$. Evidently we can assume that the energies of internal and external semiproximity of boundary $\gamma$ of the buckling region are equal. Then in (4.5') it is possible to

replace $r_0$ by $\rho$ everywhere, and we shall obtain the expression for energy of elastic deformation near the boundary of the region of buckling,

$$U=\frac{Eh}{1-\nu^2}\int_0^{2\pi}\int_0^{s^\circ}\left\{\frac{h^2}{12}[v''(\xi+u')-u''(\eta_1+v')]^2+\right.$$
$$\left.+\left[\xi u'+\eta v'+\frac{\nu u}{\rho}+\frac{u'^2+v'^2}{2}\right]^2+(1-\nu^2)\frac{u^2}{\rho^2}\rho\,ds\,d\theta.\right\} \qquad (4.5)$$

From the condition of minimum of this functional in the class of functions u and v we can determine the form of elastic deformation of shell in the vicinity of $\gamma$. Let us clarify the conditions to which u and v conform. By definition the boundary of the region of buckling $\gamma$ when $s = 0$, $u = 0$. Further, when $s = 0$, $z' = \eta + v' = 0$, i.e., $v' = -\eta$. With withdrawal from the boundary of the region of buckling u and v tend to zero, since the energy of straightening the rib is concentrated near the boundary mentioned. For investigation on the minimum of functional U it is convenient to introduce new variables instead of u, $v'$ and s:

$$\bar{u}=\frac{\nu u}{\varepsilon\rho\eta_1^2},\quad \bar{v}=\frac{v'}{\eta_1},\quad \bar{s}=\frac{\nu s}{\varepsilon\rho},\quad \varepsilon^4=\frac{\nu^4h^2}{12(1-\nu^2)\rho^2\eta_1^4}. \qquad (4.6)$$

We introduce (4.6) in (4.5) and for simplicity omit the line over u, v, s. Then (4.5) will take the form

$$U=\frac{2\pi E}{12^{\frac{3}{4}}(1-\nu^2)^{\frac{3}{4}}}h^{\frac{5}{2}}\eta_1^{\frac{5}{2}}\rho^{\frac{1}{2}}I, \qquad (4.7)$$

where

$$I=\int_0^{\bar{s}^\circ}\left\{[v'(\xi+\eta_1^2u')-\eta_1^2u''(1+v)]^2+\right.$$
$$\left.+\frac{\nu^2}{1-\nu^2}\left(u+\frac{\xi u'+v+\frac{1}{2}(\eta_1^2u'^2+v^2)}{\varepsilon}\right)^2+u^2\right\}ds.$$

Thus, the problem of determination of elastic deformations of the shell near the boundary of the region of buckling is reduced to finding functions u and v, which realize the minimum of functional I.

Subsequently we will be interested in the case of small $\alpha$ ($\alpha$ is the angle of inclination of cone generators to the plane, perpendicular to its axis). For definiteness of the problem on minimum I let us also study its limit when $h \to 0$. This gives to functional I this form:

$$I_0 = \int_0^\infty \left\{ [v_0'(\xi + \eta^2 u_0') - \eta^2 u_0'(1 + v_0)]^2 + \frac{v^2 A^2}{1 - v^2} + u_0^2 \right\} ds,$$

where $u_0$ and $v_0$ are the first terms of expansion of u and v in a series with respect to powers of parameter $\varepsilon$. A includes second terms of expansion of $u_1$ and $v_1$. They can be arranged in such a manner that A is as small as desired. Then the limiting expression for I will be

$$I_0 = \int_0^\infty \{[v_0'(\xi + \eta^2 u_0') - \eta^2 u_0'(1 + v_0)]^2 + u_0^2\} ds.$$

With $\alpha \to 0$ (and consequently, $\eta \to 0$, $\xi \to 1$) the functional will assume the form

$$I_0 = \int_0^\infty (v'^2 + u^2)\, ds. \qquad (4.8)$$

The problem of determination of main terms $u_0$ and $v_0$ of the expansion $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$, $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots$ consists of determination of the minimum of functional (4.8) with nonholonomic constraint

$$u' + v + \frac{1}{2} v^2 = 0 \qquad (4.9)$$

in the class of functions, satisfying boundary conditions,

$$u(0) = 0, \quad v(0) = 1, \quad u(\infty) = 0, \quad v(\infty) = 0.$$

The "zero" index for functions u and v is omitted for the simplicity of writing.

Let us solve the problem about the minimum $I_0$ which will enable us to obtain an explicit expression of energy of elastic deformation of the convex shell depending on parameters which determine mirror buckling. Let us set up Lagrange function $\Phi$ for this problem:

$$\Phi = v'^2 + u^2 + \lambda(s)\left(u' + v + \frac{v^2}{2}\right).$$

Euler-Lagrange equations will be:

$$\lambda(1+v) - 2v'' = 0, \quad 2u - \lambda' = 0.$$

Thus, the determination of functions u, v, communicating minimum $I_0$, is reduced to solution of the system of equations:

$$\begin{aligned} &2u - \lambda' = 0, \\ &\lambda(1+v) - 2v'' = 0, \\ &u' + v + \frac{v^2}{2} = 0, \end{aligned} \tag{4.10}$$

under conditions,

$$u(0) = 0, \quad v(0) = 1; \qquad u(\infty) = 0, \quad v(\infty) = 0.$$

Integrating the last equation from (4.10), we have,

$$u = \int_s^\infty \left(v + \frac{v^2}{2}\right) ds. \tag{4.11}$$

From the second equation (4.10) it follows that $\lambda(\infty) = 0$, since $v(\infty) = 0$, and, consequently, $v''(\infty) = 0$.

We introduce (4.11) in the first equation (4.10). Integration gives

$$\lambda = -2\int_s^\infty \int_{s_1}^\infty \left(v + \frac{v^2}{2}\right) ds_1\, ds. \tag{4.12}$$

For functions v from the second equation (4.10) we obtain integro-differential equation

$$v'' + (1+v)\int_s^\infty \int_{s_1}^\infty \left(v + \frac{v^2}{2}\right) ds_1\, ds = 0,$$

the solution of which can be sought in the form

$$v = a_1 x + a_2 y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + \ldots,$$

where $x = e^{\omega_1 s}$; $y = e^{\omega_2 s}$ and $\omega_1$, $\omega_2$ are certain complex numbers with negative real part. Determining u and v, which realize the minimum of functional $I_0$, we find that min $I_0 \approx 1.2$, $v'(0) = 1$, max $|u| \approx 0.5$. Convergence of I to $I_0$ when there is an independent tendency of h and $\alpha$ toward zero will take place, if $\overline{\varepsilon}^* \to \infty$ in integral I. For that

limit we have,

$$\bar{\varepsilon}^* = \frac{\nu\varepsilon^*}{\rho\varepsilon}, \quad \varepsilon^4 = \frac{\nu^2 h^2}{12(1-\nu^2)\rho^2\alpha^2}. \tag{4.13}$$

We assume $\varepsilon^* = \tilde{\varepsilon}^* \rho$, where $\tilde{\varepsilon}^*$ is the relative width of the zone of local bend on the boundary of buckling. In order to have $\bar{\varepsilon}^* = \frac{\nu\varepsilon^*}{\varepsilon}$ sufficiently large, it is necessary that $\varepsilon$ were sufficiently small, i.e., that $\frac{h}{\rho\alpha}$ were sufficiently small. Consequently, formula (4.7) for calculation of energy can be used, if $\frac{h}{\rho\alpha} \ll 1$.

From (4.7) it follows that the energy of elastic deformation per one unit of the length of the boundary of buckling of a conical shell is equal to:

$$\bar{U} = cEh^{\frac{5}{2}}\alpha^{\frac{5}{2}}\rho^{-\frac{1}{2}}, \quad c = \frac{I_0}{12^{\frac{3}{4}}(1-\nu^2)^{\frac{3}{4}}} \approx 0{,}12. \tag{4.14}$$

As we have shown above, in the case of any convex shell the energy of elastic deformation per one unit of the length of the boundary of buckling should be calculated by the same formula, but $\alpha$ and $\rho$ have the value,

$\alpha$ is the angle, at which the plane determining mirror buckling intersects the surface of the shell, and $\rho$ is the radius of curvature of curve $\gamma$, on which this intersection occurs. The energy of elastic deformation along all boundaries $\gamma$ of buckling is determined by formula

$$U_\gamma = \int_\gamma \bar{U}\, ds. \tag{4.15}$$

In the case of small regions of buckling with small $\alpha$ according to Meusnier's formula, we have,

$$\frac{\alpha}{\rho} \approx \frac{1}{R},$$

where $\frac{1}{R}$ is the normal curvature of the shell surface in the direction of the boundary of buckling. Hence we have that

$$\bar{U} = cE\left(\frac{h}{R}\right)^{\frac{5}{2}}\rho^2, \tag{4.16}$$

and the condition of applicability of $\left(\frac{h}{\rho\alpha} \ll 1\right)$ will be written in the form

$$\frac{Rh}{\rho^2} \ll 1.$$

Thus, in all three regions $G_1$, $G_2$, and $G_{12}$ the energy of elastic deformation of the convex shell with buckling has been found and is determined by the formula

$$U = U_0 + U_G + U_\gamma. \tag{4.17}$$

Energy $U_0$ is found by solving the problem on the elastic state of the shell in linear approximation.

Energy $U_G$ is determined by the formula

$$U_G = \frac{Eh^3}{6(1-\nu^2)} \iint_G K_\nu \, dG;$$

$$K_\nu = (k_1 + k_2)^2 - 2(1-\nu) k_1 k_2.$$

Here the region of integration is the region of mirror buckling.

Energy $U_\gamma$ is calculated according to

$$U_\gamma = \int_\gamma \bar{U} \, ds;$$

$$\left(\bar{U} = cE\left(\frac{h}{R}\right)^{\frac{5}{2}} \rho^2, \quad c = \frac{I_0}{12^{\frac{3}{4}}(1-\nu^2)^{\frac{3}{4}}}\right),$$

where $\frac{1}{R}$ is the normal curvature of shell in the direction of line $\gamma$, the boundary of mirror buckling; $\rho$ is the radius of curvature of curve $\gamma$. Integration is performed on the boundary $\gamma$ of mirror buckling. When $\nu = 0.3$ and $I_0 = 1.2$; constant $c = 0.12$.

### 3. Stresses in Elastic Supercritical Deformation of the Convex Shell

Earlier we noted that outside the vicinity of the boundary of buckling stresses in the shell are similar to those, which appear in practical deformantion with the same external load. At a sufficient

distance from the boundary of the region of buckling inside it there are bending stresses as determined by the form of mirror bucklings, which can be calculated in the following manner: mirror buckling results in the appearance on the shell surface of stretching-compression stresses, equal to:

$$\sigma_1 = \pm \frac{Eh}{1-\nu^2}(k_1 + \nu k_2), \quad \sigma_2 = \pm \frac{Eh}{1-\nu^2}(k_2 + \nu k_1). \tag{4.18}$$

On the boundary of buckling the basic bend occurs in the plane, perpendicular to the boundary.

A. V. Pogorelov [73] considers that the change of normal curvature of the shell is so big here that the initial curvature can be disregarded. We now assume that it is possible to consider the stresses originating from such bending to be similar to those in the case of the conical shell in its deformation with buckling, which was studied here earlier. Bending stresses of the conical shell in the meridian plane near the boundary of buckling are determined by the formula

$$\sigma = \pm \frac{Eh\varkappa}{2(1-\nu^2)}; \tag{4.19}$$

$$(\varkappa = v''(\xi + u') - u''(\eta + v')).$$

Passing to variables $\bar{u}$, $\bar{v}$, $\bar{s}$ (4.6) and omitting the line over them, will obtain for $\varkappa$ the expression

$$\varkappa = \frac{\alpha\nu}{\rho^2}[v''(\xi + \eta^2 u') - \eta^2 u''(1 + v)], \tag{4.20}$$

which when $h \to 0$ and $\alpha \to 0$ will be transformed into the following:

$$\varkappa = \frac{\alpha\nu}{\rho^2} v', \tag{4.21}$$

where $v'$ is the function realizing the minimum of functional $I_0$.

Introducing (4.13) in (4.21), and the latter into (4.19), we obtain,

$$\sigma = \pm \frac{12^{\frac{1}{4}} E}{2(1-\nu^2)^{\frac{3}{4}}} \rho^{-\frac{1}{2}} h^{\frac{1}{2}} \alpha^{\frac{3}{2}} v'. \tag{4.22}$$

The expression of stresses through normal curvature $\frac{1}{R}$ of the shell in the direction of the boundary of buckling $\left(\frac{1}{R} \propto \frac{\alpha}{\rho}\right)$ will assume the form:

$$\sigma = \pm c'E\frac{\rho}{R}\left(\frac{h}{R}\right)^{\frac{1}{2}}, \quad c' = -\frac{12^{\frac{1}{4}}v'}{2(1-\nu^2)^{\frac{3}{4}}}. \tag{4.23}$$

Maximum bending stresses in the vicinity of the boundary of buckling correspond to the maximum value of derivative $v'(s)$, which is attained when $s = 0$. This we can see from the fact that when $s = 0$, $v'' = 0$, and also from the second equation of the system (4.10).

When $s = 0$, $v' = 1$. Therefore, $c' = 1$ (when $\nu = 0.3$).

On the boundary of buckling of the shell the local bend is accompanied by the appearance of extension (compression) stresses of the mi. ile surface of shell in areas, perpendicular to the boundary of buckling. With small thickness these stresses can be considered to be the same as in the case of a conical shell, for which they are equal to:

$$\sigma = \pm E\varepsilon_2 = \pm\frac{u^*}{\rho}. \tag{4.24}$$

Or in variables, realizing minimum of $I_0$:

$$\sigma = \pm c''Eh^{\frac{1}{2}}\rho^{\;\frac{1}{2}}\alpha^{\frac{3}{2}}; \quad c'' = \frac{u}{12^{\frac{1}{4}}(1-\nu^2)^{\frac{1}{4}}}. \tag{4.25}$$

The minimum stress $\sigma$ will correspond to the maximum of value $u(s) \simeq 0.5$. When $\nu = 0.3$, $c'' = 0.25$. Expressing the stress through normal curvature $\frac{1}{R}$, we obtain:

$$\sigma = c''E\frac{\rho}{R}\left(\frac{h}{R}\right)^{\frac{1}{2}}. \tag{4.26}$$

In the case of small elastic bucklings of shells expressions for the energy of elastic deformation can be simplified. It is known that the form of regular, strictly convex surface in the small vicinity of point P approximates well the osculating elliptic paraboloid. Taking

the tangent plane of surface at point P for the xy plane, and main directions on the surface in this point for the direction of coordinate axes, the equation of osculating paraboloid can be written in the form

$$z=\frac{1}{2}(k_1x^2+k_2y^2).$$

Consequently, for a small height of buckling δ the region of mirror buckling with center P can be prescribed by equation

$$\delta=\frac{1}{2}(k_1x^2+k_2y^2), \tag{4.27}$$

which describes ellipse with semiaxes

$$a=\sqrt{\frac{2\delta}{k_1}},\quad b=\sqrt{\frac{2\delta}{k_2}}. \tag{4.28}$$

The area of buckling of shell will e

$$S_G=\frac{2\pi\delta}{\sqrt{k_1k_2}}. \tag{4.29}$$

The curvature of the boundary of buckling (considering x = a cos t, y = b sin t) will be written so:

$$\frac{1}{\rho}=\frac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{3}{2}}}. \tag{4.30}$$

The normal curvature of the shell surface in the direction of the boundary of buckling

$$\frac{1}{R}=\frac{2\delta}{a^2\sin^2 t+b^2\cos^2 t}. \tag{4.31}$$

In the case of small region of buckling main curvatures $k_1$ and $k_2$ change little in the region of integration and they can be considered equal to values in the center of buckling. Then (4.1) will take the form

$$U_G=\frac{Eh^3}{6(1-\nu^2)}K_\nu S_G. \tag{4.32}$$

Introducing here value $S_G$ the area of region of buckling from (4.29), we obtain,

$$U_G=\frac{2\pi\delta Eh^3K_\nu}{6(1-\nu^2)\sqrt{k_1k_2}}. \tag{4.32'}$$

For the energy of elastic deformation, connected with the local bend on the boundary of buckling,

$$U_1 = \int_1 \bar{U} ds, \quad \left[\bar{U} = cE\left(\frac{h}{R}\right)^{\frac{5}{2}} \rho^3\right]. \tag{4.15'}$$

we find, considering (4.29) and (4.30),

$$U_1 = \pi cE \frac{(2h\delta)^{\frac{5}{2}}}{a^2 b^2}(a^2 + b^2), \tag{4.15''}$$

or, considering (4.28):

$$U_1 = \pi cE (2\delta)^{\frac{3}{2}} h^{\frac{5}{2}} (k_1 + k_2), \tag{4.33}$$

which can be written through mean curvature $K = \frac{1}{2}(k_1 + k_2)$ of the shell in the center of buckling, in this manner:

$$U_1 = 2\pi cE (2\delta)^{\frac{3}{2}} h^{\frac{5}{2}} K; \quad (c \approx 0{,}12). \tag{4.34}$$

This formula can be used under the condition that

$$\frac{h}{2\delta}\left(\frac{k_1}{k_2}\right) \ll 1; \quad (k_1 > k_2). \tag{4.35}$$

Formula (4.23) for stresses in the case of small region of buckling, after introduction in it of values $\frac{1}{\rho}$ and $\frac{1}{R}$ from (4.30) and (4.31), will take the form

$$\sigma = \pm \frac{c'E (2\delta)^{\frac{3}{2}} h^{\frac{1}{2}}}{ab}, \tag{4.36}$$

or, taking (4.28) into consideration,

$$\sigma = \pm c'E (2\delta)^{\frac{1}{2}} h^{\frac{1}{2}} \sqrt{k_1 k_2}. \tag{4.37}$$

Let us note that stresses along the boundary of bucklings are constant.

For extension (compression) stresses of the middle surface, caused by local flexure, formula (4.25) was set up which in case of small region of buckling will take the form

$$\sigma = \pm c''E (2\delta)^{\frac{1}{2}} h^{\frac{1}{2}} \sqrt{k_1 k_2}. \tag{4.38}$$

## 4. The Stability of Elastic Equilibrium of a Convex Shell with Buckling

Let us assume that the load inside the region of small buckling is constant. The condition of equilibrium of the elastic state of shell can be written as an equality of work dA of external forces and change of the energy of elastic deformation:

$$dA = dU_1 + dU_2. \tag{4.39}$$

For determination of parameters of elastic deformation of the shell (center and height of buckling) we will find the elementary work dA depending on them. Work dA, produced by the external load q in the transition from buckling 1-1 (Fig. 65) to infinitely similar buckling 2-2, is equal to qdv, where dv is the change of volume caused by buckling. Consequently, dA is the differential of function A = qv, where v is the double volume of the segment, cut off by the plane which determines the mirror buckling of the shell.

Fig. 65.

Let us find the volume of this segment. We have,

$$\frac{v}{2} = \int_0^b S(z)\,dz,$$

where S(z) is the area of section, parallel to the base of segment at a distance z from the summit. Since the form of the section is similar to an ellipse with semiaxes

$$a(z) = \sqrt{\frac{2z}{k_1}}, \quad b(z) = \sqrt{\frac{2z}{k_2}},$$

then

$$S(z) = \frac{2\pi z}{\sqrt{k_1 k_2}}.$$

Hence

$$\frac{v}{2} = \int_0^b \frac{2\pi z}{\sqrt{k_1 k_2}}\,dz = \frac{\pi b^2}{\sqrt{k_1 k_2}}, \tag{4.40}$$

and, consequently

$$A = \frac{2\pi b^2}{\sqrt{k_1 k_2}}. \tag{4.41}$$

Let us assume that the position of the center of buckling is known. Then the height of buckling will be determined from the relationship

$$\frac{d}{d\delta}(A-U_{\gamma}-U_{G})=0. \tag{4.42}$$

We introduce the load parameter

$$\xi=\frac{q}{Eh^{2}k_{1}k_{2}} \tag{4.43}$$

and the parameter of height of buckling

$$\eta=\sqrt{\frac{2\delta}{h}}. \tag{4.44}$$

Then formulas (4.41), (4.34), (4.31) for A, $U_{\gamma}$ and $U_{G}$, as well as (4.42) will be transformed into the following formulas:

$$A=\frac{\pi}{2}E\xi\eta^{4}h^{4}\sqrt{k_{1}k_{2}}, \tag{4.45}$$

$$U_{\gamma}=2\pi cE\eta^{3}h^{4}\frac{1}{2}(k_{1}+k_{2}), \tag{4.46}$$

$$U_{G}=\frac{\pi E\eta^{2}h^{4}K_{\nu}}{6(1-\nu^{2})\sqrt{k_{1}k_{2}}}, \tag{4.47}$$

$$\frac{d}{d\eta}(A-U_{\gamma}-U_{G})=0, \tag{4.48}$$

according to which we can easily set up the equilibrium equation, determining the height of buckling,

$$\xi\eta^{2}-3c\eta\overline{K}-\frac{\overline{K}_{\nu}}{6(1-\nu^{2})}=0, \tag{4.49}$$

where

$$\overline{K}=\frac{K}{\sqrt{\Gamma}},\quad \overline{K}_{\nu}=\frac{K_{\nu}}{\sqrt{\Gamma}}=4\overline{K}^{2}-2(1-\nu),$$

$$\left(\Gamma=k_{1}k_{2};\quad K=\frac{1}{2}(k_{1}+k_{2})\right).$$

After determining $\eta$, we find the height of buckling $\delta=\frac{1}{2}h\eta^{2}$, and then by formulas (4.28) — the dimensions of the region of buckling,

$$a=\sqrt{\frac{2\delta}{k_{1}}},\quad b=\sqrt{\frac{2\delta}{k_{2}}}. \tag{4.28}$$

The position of the center of buckling P can be determined from the following considerations. The "height" of buckling $\eta$, determined from

(4.49), will be the function of P. To the true buckling corresponds such position of P, with which the energy of elastic deformation $U_\gamma + U_G$ will be maximum. Introducing $\eta$ from (4.49) into the expression of energy of elastic deformation $U_\gamma + U_G$, considered as a function of P, we find the position of the center of buckling from the conditions of maximum of this expression. The same result can be obtained from the condition of maximum of expression

$$A = \frac{\pi}{2} E\xi\eta^4 h^4 \sqrt{k_1 k_2}, \tag{4.50}$$

in which we must introduce $\eta$, determined from equation (4.49). In the case of action on the shell of a concentrated force F, the point of application of which can be considered the center of buckling P, the elementary work $dA = 2F d\delta$, where $\delta$ is the height of the segment of mirror buckling. Assuming that

$\delta = \frac{k\eta^2}{2}$ and $F = \frac{\pi}{2} E\xi h^3 \sqrt{k_1 k_2}$, we obtain

$$A = \pi E\xi\eta^3 h^4 \sqrt{k_1 k_2}. \tag{4.51}$$

From condition (4.42), taking into consideration (4.46) and (4.47), we obtain the equation, which is satisfied by parameter $\eta$, determining the height of buckling $\delta$:

$$\xi - 3c\eta\bar{K} - \frac{\bar{K}_\nu}{6(1-\nu^2)} = 0, \tag{4.52}$$

and hence

$$\eta = \frac{1}{3c}\left(\xi - \frac{\bar{K}_\nu}{6(1-\nu^2)}\right). \tag{4.53}$$

If an arbitrary load q acts on the shell and the load is concentrated inside the region of buckling (near the boundary of buckling q = 0), then its action is equivalent to the resultant concentrated load F. In this case $\eta$ is determined by the formula (4.53).

When a load, depending on the form of surface, acts on the shell, with a given position of the center of buckling P the acting load will

be a known function $q(P, \delta)$ of the height of buckling $\delta$. Consequently, we can find the work of external forces $A(P, \delta)$, and the height $\delta(P)$ from the condition of equilibrium (4.42). The position of the center of buckling is determined from the condition of maximum $A[P, \delta(P)]$ as a function of P.

In the case of action of impact load (impact at point P) we may assume that the energy of impact T is transferred in full into the energy of elastic deformation, connected with buckling ($U_\gamma + U_G$), since the appearance of buckling lowers rigidity of the shell as an elastic system. The height of buckling is determined from equality

$$T = U_\gamma + U_G.$$

For example, in the case of buckling of a spherical shell with the radius R and thickness h acted upon by the concentrated force F we have,

$$\zeta = \frac{FR}{\pi E h^3}, \ \bar{K}_\nu = 2(1+\nu),$$

$$\eta = \frac{1}{3c}\left(\frac{FR}{\pi E h^3} - \frac{1}{3(1-\nu)}\right), \quad \delta = \frac{h\gamma_1^2}{2}.$$

Let us consider the state of elastic equilibrium to be stable, if

$$d^2(A - U_\gamma - U_G) < 0, \qquad (4.54)$$

and unstable, if

$$d^2(A - U_\gamma - U_G) > 0. \qquad (4.55)$$

Let us assume that a continuous load q acts on the shell, such a load that A, $U_\gamma$ and $U_G$ are described by formulas (4.45)-(4.47). We have,

$$\frac{d}{d\delta}(A - U_\gamma - U_G) = \frac{d}{d\gamma_1}(A - U_\gamma - U_G)\frac{d\gamma_1}{d\delta} = 0.$$

Hence

$$\frac{d^2}{d\delta^2}(A - U_\gamma - U_G) = \frac{d^2}{d\gamma_1^2}(A - U_\gamma - U_G)\left(\frac{d\gamma_1}{d\delta}\right)^2. \qquad (4.56)$$

Thus, the question of stability of equilibrium is resolved depending on the sign of expression

$$\frac{d^2}{d\gamma_1^2}(A - U_\gamma - U_G) = \pi E h^4\left(6\xi\gamma_1^2 - 12c\gamma_1\bar{K} - \frac{1}{3}\frac{\bar{K}_\nu}{1-\nu^2}\right). \qquad (4.57)$$

Taking the condition (4.48) of equilibrium of shell into consideration, we obtain

$$\frac{d^2}{d\eta^2}(A - U_1' - U_G) = \pi E h^4 \sqrt{\Gamma}\left(6c\eta\bar{K} + \frac{2\bar{K}_\nu}{3(1-\nu^2)}\right),$$

from which it is clear that

$$\frac{d^2}{d\eta^2}(A - U_1 - U_G) > 0.$$

Consequently, all states of elastic equilibrium of the convex shell with buckling acted upon by a continuous load, are unstable.

The instability of the elastic state of shell in buckling is the cause of "snapping" of the shell, i.e., intermittent buckling without any increase of the external load.

During the action of concentrated force F:

$$\frac{d^2}{d\eta_1^2}(A - U_1 - U_G) = \pi E h^4 \sqrt{k_1 k_2}\left(2\xi - 12c\eta_1\bar{K} - \frac{\bar{K}_\nu}{3(1-\nu^2)}\right), \qquad (4.58)$$

or, taking the condition of equilibrium (4.52) into consideration, we obtain

$$\frac{d^2}{d\eta_1^2}(A - U_1 - U_G) = -6\pi c E h^4 \eta \bar{K}\sqrt{k_1 k_2} < 0,$$

i.e., elastic states of the equilibrium of shell with buckling under the action of concentrated force are stable.

## 5. Upper and Lower Critical Loads

The upper critical load cannot be determined within the scope of A. V. Pogorelov's syntheses, since they pertain to the region of deformations with significant buckling. However, we can express the ideas on the effect of compulsory buckling, for example, the initial bending, on the value of upper critical load. Let us assume that the height of buckling of the shell, on which load q is acting, will be $\delta$. Formula (4.49) gives us the relation $q(\delta)$ in variables $\xi$, $\eta$. Since elastic states of equilibrium under the action of continuous load are unstable, then with $q < q(\delta)$ the buckling disappears, and with

$q > q(\delta)$ — conversely, the buckling increases and popping of shell occurs. Thus, in buckling of shell to the height $\delta$ the upper critical load is lowered at least to the value $q(\delta)$. As an example of application of this result will consider the problem of designing such a shell that the possibility of snapping of the shell within the frames under the action of a given load was excluded. For the determination of distance between the frames we must determine dimensions of the region of bucklings, satisfying the given load, and after that dispose the frames in such a manner that not one of the mentioned regions could be placed between them. With such location of reinforcing elements any buckling, whatever its cause cannot develop under the action of load q, and consequently, disappears.

The lower critical load in the continuous loading of shell will be the lower boundary of loads q, satisfying the stable states of elastic equilibrium with buckling. From the point of view the theory presented here such a conception of the lower critical load has no meaning, since all elastic states of convex shell with buckling are unstable.

Let us consider the spherical segment, fastened on edges, subjected to the action of continuously increasing external pressure q. Under a certain pressure q the shell will pop, after which we shall continuously decrease the load. The region of buckling hardly changes until the pressure is lowered to a certain value $q_l < q_u$, when the reverse snapping of the shell will occur. The value $q_l$ is taken for the lower critical load. On the samples, subjected to such tests, traces of plastic deformations were observed. According to the theory of elastic state of shell with buckling expounded here, the character of the course of such an experiment is explained in the following manner. The buckling of shell, which began under the load $q_u$ with the

unlimited elasticity of the shell material, cannot stop, since the work of external forces increases faster than the energy of elastic deformation connected with buckling. But in the case of appearance of plastic deformation on the boundary of buckling, further buckling is associated with a large energy consumption, which is not replenished by the work of external forces, and, consequently, the buckling stops. Thus, the determination of lower critical load $q_1$ for strictly convex shells is impossible without an estimate of plastic deformations. Within the limits of elastic deformations for $q_1$ we can give an estimate from above. Indeed, in view of the instability of the elastic state of shell with buckling under the action of continuous load, snapping of the shell can be produced from any such state by any small perturbation. The critical load $q_1$ does not exceed the lower boundary of loads q, which satisfy elastic states with buckling. Let us assume that load $q^*$ corresponds to the appearance on the boundary of buckling of nonelastic deformations and corresponding stresses, which we consider to be equal to the yield point $\sigma_s$. We find load $q^*$ from the following considerations.

Height $\delta$ of buckling of the shell, with which on the boundary of buckling of shell local bending stresses appear, which are determined according to (4.37) and equal in our case to $\sigma_s$, can be determined from the condition

$$\sigma_s = c'E(2\delta)^{\frac{1}{2}}h^{\frac{1}{2}}\sqrt{k_1k_2}. \qquad (4.59)$$

Assuming $\delta = \frac{1}{2}h\eta^2$, we obtain

$$\sigma_s = c'Eh\eta\sqrt{k_1k_2}. \qquad (4.60)$$

Introducing in (4.49) value $\eta$, expressed through $\sigma_s$, we find the value of parameter of load $\xi$, satisfying the load $q^*$ sought (let us remember that $q = \xi Eh^2k_1k_2$):

$$\xi^* = 3\alpha c' (Kh)\frac{E}{\sigma_s} + \frac{c'^2}{6(1-\nu^2)}[4(Kh)^2 - 2(1-\nu)\Gamma h^2]\left(\frac{E}{\sigma_s}\right)^2 \qquad (4.61)$$

$\left(K = \frac{1}{2}(k_1 + k_2)\right.$ is the average curvature and $\Gamma = k_1 k_2$ is the Gaussian curvature in the center of buckling).

In particular, for the spherical shell with a radius R

$$\xi^* = 3\alpha c'\left(\frac{h}{R}\right)\frac{E}{\sigma_s} + \frac{c'^2}{3(1-\nu^2)}\left(\frac{h}{R}\right)^2\left(\frac{E}{\sigma_s}\right)^2. \qquad (4.62)$$

Thus, the experimentally determined lower critical load $q_1 = \xi_1 Eh^2 k_1 k_2$ does not exceed $q^* = \xi^* Eh^2 k_1 k_2$, where $\xi^*$ is from (4.61). This conclusion can be drawn, if the considered deformations are in the region of permissible theories, i.e., we should have

$$\frac{h}{2\delta}\left(\frac{k_1}{k_2}\right) \ll 1,$$

from which it follows that formula (4.61) can be used, if

$$\sigma_s \gg Ehk_1, \qquad (4.63)$$

where $k_1$ is largest of the main curvatures. Let us note that the requirement (4.63) can be [73] weakened to $\sigma_s > Ehk_1$.

### 6. A. V. Pogorelov's Simplified Theory for the Supercritical Elastic State of Strictly Convex Shells

If in the energy expression of elastic deformation we disregard the bending energy of the shell in the region of buckling, then the theory of elastic state of convex shells in buckling is essentially simplified. At the same time such an assumption is possible on the basis that the bending energy of a shell in the region of buckling has a subordinate value in comparison with the energy, produced by the local bend on the boundary of buckling.

Let us take for simplicity a spherical shell. We have,

$$U_G = \frac{2\pi\delta Eh^3}{6(1-\nu)R}, \quad U_T = 2\pi c E (2\delta)^{\frac{3}{2}} h^{\frac{5}{2}} \frac{1}{R}. \qquad (4.64)$$

Introducing in these formulas the radius $\rho$ of the circle of buckling, instead of $\delta$ we obtain,

$$U_G = \frac{\pi E \rho^2 h^3}{6(1-\nu)R^2}, \quad U_\gamma = 2\pi c E \rho^3 \left(\frac{h}{R}\right)^{\frac{5}{2}}. \tag{4.65}$$

Hence

$$\frac{U_G}{U_\gamma} = \frac{1}{12(1-\nu)c}\sqrt{\frac{Rh}{\rho^2}} \simeq \sqrt{\frac{Rh}{\rho^2}}. \tag{4.66}$$

Formula for $U_\gamma$ is derived in the assumption that $\frac{Rh}{\rho^2} \ll 1$. Consequently, the theory pertains to such deformations with buckling, with which $U_G \ll U_\gamma$. Therefore, it is permissible to disregard term $U_G$ in the energy expression of elastic deformation, considering its equal simply to the energy of local bend $U_\gamma$ on the boundary of buckling. Qualitatively the same result is obtained for an arbitrary convex shell. Further, everywhere here we will consider that the energy of elastic deformation of shell consists only of the energy of local bend on the boundary of buckling.

Let us now determine the state of equilibrium of shell with buckling for different loading methods.

In the case of buckling of the shell under the action of a continuous load, as we did earlier, we characterize the load q acting on the shell with parameter $\xi = \frac{q}{Eh^2\Gamma}$, and the height of buckling $\delta$ — with the parameter $\eta = \sqrt{\frac{2\delta}{h}}$. Then we obtain the following simple relationship between $\xi$ and $\eta$:

$$\xi\eta - 3c\frac{K}{\sqrt{\Gamma}} \approx 0. \tag{4.67}$$

Hence we find the height of buckling $\delta$ depending on the active load q,

$$\delta = \frac{9c^2E^2h^5\Gamma K^2}{2q^2}. \tag{4.68}$$

The position of the center of buckling is determined from the condition of steadiness of the energy of elastic deformation,

$$U_{\gamma}=2\pi cE(2\delta)^{\frac{3}{2}}h^{\frac{5}{2}}K, \qquad (4.69)$$

where instead of δ we must place the expression, determined by formula (4.68).

In the case of buckling of the shell under the action of a concentrated force we characterize effective load F with parameter ξ, determined by formula

$$F=\frac{\pi}{2}\xi Eh^{2}\sqrt{k_1k_2}. \qquad (4.70)$$

Then the condition of equilibrium of shell will be obtained in form

$$\xi-3c\eta_1\frac{K}{\sqrt{k_1k_2}}=0. \qquad (4.71)$$

Hence, for the height of buckling δ under the action of concentrated force F we obtain formula

$$\delta=\frac{4F^{2}}{9c^{2}\pi^{2}E^{2}h^{4}K^{2}}. \qquad (4.72)$$

In the case of buckling of a shell under the action of an impact load the concentrated impact on the shell communicating energy A, produces buckling δ, determined from the relationship $A = U_{\gamma}$, i.e.,

$$A=2\pi cE(2\delta)^{\frac{3}{2}}h^{\frac{5}{2}}K, \qquad (4.73)$$

where all values pertain to the point of the shell's surface, where the impact occurred. Thus,

$$\delta=\frac{1}{2}\left(\frac{A}{2\pi cEh^{\frac{5}{2}}K}\right)^{\frac{2}{3}}. \qquad (4.74)$$

In all cases of loading, maximum stresses appearing from local bending on the boundary of buckling, are determined by the formula

$$\sigma=\pm c'E(2\delta)^{\frac{1}{2}}h^{\frac{1}{2}}\sqrt{k_1\cdot k_2}\quad (c'\simeq 1). \qquad (4.75)$$

Let us consider the problem of critical loads.

The upper critical load. As it was shown earlier, the compulsory buckling of shell to the height δ lowers the upper critical load at

least to the value q(δ), which is established from the condition of equilibrium of the shell under the action of a continuous load, i.e., from the relationship (4.68), where all values pertain to the center of buckling.

Hence it follows that the convex shell, being under the action of a continuous load q and concentrated force F, satisfying the inequality

$$\frac{4F^3}{9c^4\pi^3E^3h^6K^3} > \frac{9c^6E^3h^9\Gamma K}{2q^6}, \tag{4.76}$$

cannot fail to snap.

We conclude analogously that if the shell, being under action of a continuous load q, receives an impact, communicating energy A, and the condition

$$\frac{1}{2}\left(\frac{A}{2\pi cEh^{\frac{5}{2}}K}\right)^{\frac{2}{3}} > \frac{9c^6E^4h^9\Gamma K^3}{2q^6}. \tag{4.77}$$

is met, then the shell snaps.

## 7. The Lower Critical Load for Sloping Convex Shells

A convex shell is termed a sloping* shell if in any buckling of it plastic deformations do not appear. We explain this on the example of a spherical segment.

Let the spherical segment with curvature $\frac{1}{R}$ have the height $\delta_0$. Any buckling of such a segment has the height $\delta < \delta_0$. The requirement that the segment be sloping, is that with any $\delta < \delta_0$

$$C'E(2\delta)^{\frac{1}{2}}h^{\frac{1}{2}}\frac{1}{R} < \sigma_s. \tag{4.78}$$

In other words, in any buckling to the height $\delta < \delta_0$, the maximum stresses appear in the shell material, would be less than the yield

*See classification of shells of M. A. Koltunov in § 2, Chapter V.

point and, consequently, would not produce plastic deformations. Let us assume that the sloping spherical segment is fastened on the edge.

In Fig. 66 we depict graphs of functions $\frac{\partial A}{\partial \delta}$ and $\frac{\partial U}{\partial \delta}$, where A is the work, produced by the external load in buckling, and U is the energy of deformation. Points of intersection 1 and 2 on these graphs correspond to states of equilibrium of the segment with buckling. Namely at point 1 — unstable equilibrium, and at point 2 — stable. Graph C depicts function $\frac{\partial A}{\partial \delta}$, corresponding to the lower critical load $q_1$.

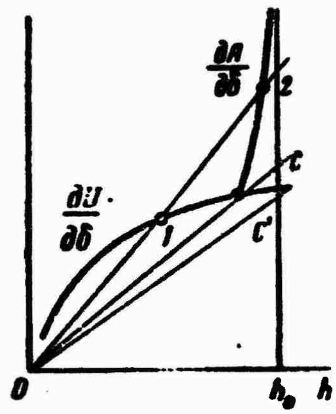


Fig. 66.

In order to determine the lower critical load, it is sufficient to know straight line C. In connection with the determination of straight line C we note the following. Up to values δ, sufficiently close to $\delta_0$, $\frac{\partial U}{\partial \delta} = C_1 \sqrt{\delta}$. For values δ, very close to $\delta_0$, $\frac{\partial U}{\partial \delta}$ increases strongly. We will not made a large error, if instead of C we take straight line C' passing through the point of intersection of straight line $\delta = \delta_0$ with parabola $\frac{\partial U}{\partial \delta} = C\sqrt{\delta}$. Along the straight line C' we find load q, close to critical $q_1$.

The analytically adduced consideration corresponds to the definition of the lower critical load as a load, balancing buckling to the height $\delta_0$, and, consequently, is found from the relationship

$$\delta_0 = \frac{9C^2E^2h^5}{2q_н^2R^4}. \text{ [н = 1 = lower]} \qquad (4.79)$$

Thus, for the sloping segment, fastened on the edge, the lower critical load

$$q_н = \left(\frac{9C^2E^2h^5}{2\delta_0 R^4}\right)^{\frac{1}{2}}. \qquad (4.80)$$

If instead of the height of segment $\delta_0$ we introduce the radius of base $\rho$, then this formula takes form

$$q_n = 3CE\frac{h}{\rho}\left(\frac{h}{R}\right)^{\frac{3}{2}}. \qquad (4.81)$$

Analogous consideration can serve as the basis of determination of the lower critical load for an arbitrary sloping convex shell. Let us adduce the final result.

Let us assume that the convex sloping shell with little variable average K and Gaussian Γ curvature is fastened on the edge. Let us assume that $\delta_0$ is the maximum height of the segment, which can be cut off from the shell by a plane, not intersecting its edge. Then the lower critical load for such shell is determined from the relationship

$$\delta_0 = \frac{9C^2E^2h^2\Gamma K^2}{2q_n^2}. \qquad (4.82)$$

The earlier received estimate for the lower critical load $q_1$ now, is now simplified when we disregarded the bending energy in the region of buckling. Namely,

$$q_n < 3CC'E(\Gamma h^2)(Kh)\frac{E}{\sigma_s}. \qquad (4.83)$$

In particular, for the spherical shell of radius R and thickness h:

$$q_n < 3CC'E\left(\frac{h}{R}\right)^3\frac{E}{\sigma_s}. \qquad (4.84)$$

We now face the problem, whether the above mentioned estimate has a value, close to the lower critical load? We can answer that in the case of a clearly expressed yield point for the shell material the estimate is close to the lower critical load. Let us explain this assumption.

Let us assume that constitution diagram of the shell material is close to ideal plasticity (Fig. 67). We turn to the graphic portrayal of relationships $\frac{\partial A}{\partial \delta}$ and $\frac{\partial U}{\partial \delta}$ (Fig. 68). Point E in the graph $\frac{\partial U}{\partial \delta}$

corresponds to the moment of appearance of plastic deformations on the boundary of buckling. To point E the graph $\frac{\partial U}{\partial \delta}$ is a parabola, beyond this point it sharply rises upwards because of plastic deformations.

Straight line C determines the lower critical load, but straight line $C_E$ — determines the estimate obtained for it. To have these straight lines near one another, it is necessary that the energy of deformation increase sharply with the appearance of plastic deformations. And this will be the case when the shell material has a clearly expressed yield point.

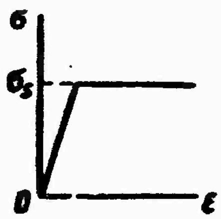

Fig. 67.

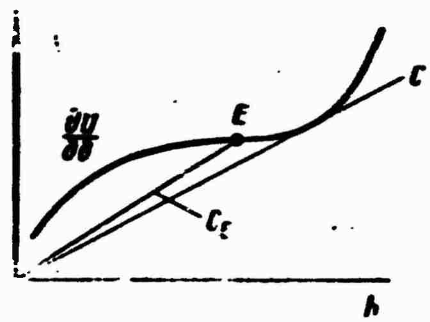

Fig. 68.

In conclusion let us note that the formula for the lower critical load $q_1$ can only be used for relatively thin shells. Namely, we should have

$$\sigma_s > Ehk_1, \qquad (4.85)$$

where $k_1$ is the largest of the main curvatures. In particular, for a spherical shell

$$\frac{h}{R} < \frac{\sigma_s}{E}. \qquad (4.86)$$

## CHAPTER VII

## SPECIAL PROBLEMS IN CALCULATION OF SHELLS

### § 1. Variational Formulation of the Problem on the Elastoplastic Deformation of Shells

Stresses, appearing in a body during active elastoplastic deformation, have the potential, representing the work of internal forces [2]. In the case of incompressible material the work of internal forces, performed per one unit of volume, is equal to:

$$W = \int_0^{e_i} \sigma_i de_i.$$

It is admissible to expect that forces T and moments M, appearing in the shell, also have a potential, which presents the work of internal forces, acting per one unit of area of the middle surface,

$$U = \int_{-\frac{h}{2}}^{\frac{h}{2}} W dz.$$

The variation of function U, corresponding to the variations of deformations $\delta\varepsilon_1$, $\delta\varepsilon_2$, $\delta\varepsilon_{12}$ and distortions $\delta\varkappa_1$, $\delta\varkappa_2$, $\delta\varkappa_{12}$, should be equal to the work of forces $T_1$, $T_2$, $T_{12}$ and moments $M_1$, $M_2$, $M_{12}$ on variations of deformations and distortions,

$$\delta U = T_1\delta\varepsilon_1 + T_2\delta\varepsilon_2 + 2T_{12}\delta\varepsilon_{12} - M_1\delta\varkappa_1 - M_2\delta\varkappa_2 - 2M_{12}\delta\varkappa_{12}. \tag{1.1}$$

After calculating variation $\delta U$ by the formula

$$\delta U = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i \delta e_i \, dz, \tag{1.2}$$

where

$$e_i = \frac{2}{\sqrt{3}} \sqrt{P_\varepsilon - 2zP_{\varepsilon\varkappa} + z^2 P_\varkappa}, \tag{1.3}$$

$$P_\varepsilon = \varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2^2 + \varepsilon_{12}^2,$$

$$P_\varkappa = \varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2 + \varkappa_{12}^2, \tag{1.4}$$

$$P_{\varepsilon\varkappa} = \varepsilon_1\varkappa_1 + \varepsilon_2\varkappa_2 + \frac{1}{2}\varepsilon_1\varkappa_2 + \frac{1}{2}\varepsilon_2\varkappa_1 + \varepsilon_{12}\varkappa_{12},$$

we have the method of expression of forces and moments through deformations and distortions

$$T_1 = \frac{\partial U}{\partial \varepsilon_1}, \quad T_2 = \frac{\partial U}{\partial \varepsilon_2}, \quad T_{12} = \frac{1}{2}\frac{\partial U}{\partial \varepsilon_{12}},$$
$$M_1 = -\frac{\partial U}{\partial \varkappa_1}, \quad M_2 = -\frac{\partial U}{\partial \varkappa_2}, \quad M_{12} = -\frac{1}{2}\frac{\partial U}{\partial \varkappa_{12}}. \tag{1.5}$$

Now we will consider the possibility of formulating the problem without equations of equilibrium of the element in the form of variational equation of equilibrium of the shell. For that purpose it is necessary to set up a variation of work of internal forces of the entire shell,

$$\delta V = \iint \delta U \, d\Sigma, \tag{1.6}$$

where the integral is distributed throughout the entire middle surface $\Sigma$, and $\delta U$ has the expression (1.1), or

$$\delta U = \frac{2}{3}\delta P_\varepsilon J_1 - \frac{4}{3}\delta P_{\varepsilon\varkappa} J_2 + \frac{2}{3}\delta P_\varkappa J_3. \tag{1.7}$$

Here,

$$J_1 = \frac{\sqrt{3}}{2P_\varkappa^{\frac{1}{2}}} B,$$

$$J_2 = \frac{\sqrt{3}P_{\varepsilon\varkappa}}{2P_\varkappa^{\frac{3}{2}}} B + \frac{3}{4P_\varkappa} A, \tag{1.8}$$

$$J_3 = \frac{3\sqrt{3}}{8P_\varkappa^{\frac{3}{2}}} C + \frac{\sqrt{3}P_{\varepsilon\varkappa}^2}{2P_\varkappa^{\frac{5}{2}}} B + \frac{3P_{\varepsilon\varkappa}}{2P_\varkappa^2} A.$$

Here if the flexural strain predominates

$$\left(-\frac{h}{2} < z_0 = \frac{P_{z1}}{P_z} < \frac{h}{2}\right),$$

then we add index "0" to values A, B, C and calculate them by the formulas:

$$A_0 = -\int_{e_{i0}}^{e_{i1}} \sigma_i de_i + \int_{e_{i0}}^{e_{i2}} \sigma_i de_i = \int_{e_{i1}}^{e_{i2}} \sigma_i de_i,$$

$$B_0 = \int_{e_{i0}}^{e_{i1}} \frac{\sigma_i de_i}{\sqrt{e_i^2 - e_{i0}^2}} + \int_{e_{i0}}^{e_{i2}} \frac{\sigma_i de_i}{\sqrt{e_i^2 - e_{i0}^2}}, \tag{1.9}$$

$$C_0 = \int_{e_{i0}}^{e_{i1}} \sigma_i \sqrt{e_i^2 - e_{i0}^2}\, de_i + \int_{e_{i0}}^{e_{i2}} \sigma_i \sqrt{e_i^2 - e_{i0}^2}\, de_i,$$

if however, extension — compression of the middle surface $\left(|z_0| > \frac{h}{2}\right)$, dominates, then to values A, B, C we ascribe index "1" and calculate them by the formulas,

$$A_1 = A_0 = \int_{e_{i1}}^{e_{i2}} \sigma_i de_i,$$

$$B_1 = \int_{e_{i1}}^{e_{i2}} \frac{\sigma_i de_i}{\sqrt{e_i^2 - e_{i0}^2}} \operatorname{sign}(e_{i2} - e_{i1}), \tag{1.10}$$

$$C_1 = \int_{e_{i1}}^{e_{i2}} \sigma_i \sqrt{e_i^2 - e_{i0}^2}\, de_i \operatorname{sign}(e_{i2} - e_{i1}).$$

Let us note that point $z = z_0$ is the point of minimum $e_i$, since $\frac{d^2 e_i}{dz^2} > 0$, and, consequently, inequalities:

$$e_{i1} > e_{i0}, \quad e_{i2} > e_{i0}, \tag{1.11}$$

always take place; here we adopted designation,

$$
\begin{aligned}
e_{i1} &= \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon + hP_{\varepsilon\varkappa} + \frac{h^2}{4}P_\varkappa} \quad \left(z = -\frac{h}{2}\right), \\
e_{i2} &= \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon - hP_{\varepsilon\varkappa} + \frac{h^2}{4}P_\varkappa} \quad \left(z = +\frac{h}{2}\right), \\
e_{i0} &= \frac{2}{\sqrt{3}\sqrt{P_\varkappa}}\sqrt{P_\varkappa P_\varepsilon - P_{\varepsilon\varkappa}^2} \quad \left(z = z_0 = \frac{P_{\varepsilon\varkappa}}{P_\varkappa}\right);
\end{aligned}
\tag{1.12}
$$

these formulas give values of intensity of deformations in three points, located on axis z.

Then it is necessary to set up the work of surface forces on variations of displacements $\delta u$, $\delta v$, $\delta w$ and the work of generalized forces of the edge on their corresponding variations of generalized displacements, made up of $\delta u$, $\delta v$, $\delta w$ and their derivatives with respect to coordinates; designating this work through $\delta' A$, we obtain the variational equation of equilibrium

$$\delta V = \delta' A,$$

the solution of which can be sought, for instance, by the Ritz method.

## § 2. A. A. Il'yushin's Final Relationship

Let us show that between forces and moments there exists a final (not differential) relationship, which was found by A. A. Il'yushin [76].

If the intensity of deformations $e_i$ of any layer of the shell is sufficiently great as compared to the yield point $e_s$, i.e.

$$\frac{2}{\sqrt{3}}\sqrt{P_\varepsilon - 2zP_{\varepsilon\varkappa} + z^2 P_\varkappa} = e_i \gg e_s, \tag{2.1}$$

and its material is not strengthened, then the law $\sigma_i = \Phi(e_i)$ coincides with Von Mises' condition of plasticity

$$\sigma_i = \sigma_s = \text{const}, \tag{2.2}$$

or can be approximately replaced by St. Venant-Coulomb condition of plasticity,

$$\tau_m = \frac{\sigma_s}{\sqrt{3}} = \text{const}. \tag{2.3}$$

In this case there exists a final relationship between forces and moments.

Actually, according to formulas (1.9) and (1.10), taking out of the integral the constant $\sigma_i$, we can calculate values of functions A, B, C. Thus, in the case of dominating flexural strains, formulas (1.9) take the form,

$$A_0 = \sigma_s (e_{i2} - e_{i1}),$$

$$B_0 = \sigma_s \ln \frac{(e_{i1} + \sqrt{e_{i1}^2 - e_{i0}^2})(e_{i2} + \sqrt{e_{i2}^2 - e_{i0}^2})}{e_{i0}^2}, \tag{2.4}$$

$$C_0 = \frac{\sigma_s}{2} \left(e_{i1} \sqrt{e_{i1}^2 - e_{i0}^2} + e_{i2} \sqrt{e_{i2}^2 - e_{i0}^2}\right) - \frac{1}{2} e_{i0}^2 B_0. \tag{2.5}$$

and in the case of dominating elongations of the middle surface from formulas (1.10) we find,

$$A_1 = \sigma_s (e_{i2} - e_{i1}),$$

$$B_1 = \sigma_s \left| \ln \frac{e_{i2} + \sqrt{e_{i2}^2 - e_{i0}^2}}{e_{i1}^2 + \sqrt{e_{i1}^2 - e_{i0}^2}} \right|, \tag{2.6}$$

$$C_1 = \frac{\sigma_s}{2} \left| e_{i2} \sqrt{e_{i2}^2 - e_{i0}^2} - e_{i1} \sqrt{e_{i1}^2 - e_{i0}} \right| - \frac{e_{i0}^2}{2} B_0.$$

In both cases values $e_{i1}$, $e_{i2}$, $e_{i0}$ are expressed by formulas (1.12). Considering the latter as equations with respect to the three quadratic forms $P_\varepsilon$, $P_{\varepsilon\varkappa}$, $P_\varkappa$, we copy them in the form,

$$P_\varepsilon + hP_{\varepsilon\varkappa} + \frac{h^2}{4} P_\varkappa = \frac{3}{4} e_{i1}^2,$$

$$P_\varepsilon - hP_{\varepsilon\varkappa} + \frac{h^2}{4} P_\varkappa = \frac{3}{4} e_{i2}^2,$$

$$P_\varepsilon P_\varkappa - P_{\varepsilon\varkappa}^2 = \frac{3}{4} e_{i0}^2 P_\varkappa.$$

The solution of these equations with respect to quadratic forms has the following results,

$$
\begin{gathered}
hP_{\varepsilon\varkappa} = \frac{3}{8}(e_{i1}^2 - e_{i2}^2),\\
P_\varepsilon = \frac{3}{8}(e_{i1}^2 - e_{i2}^2) - \frac{h^2}{4}P_\varkappa,\\
\frac{h^2}{4}P_\varkappa = \frac{3}{16}\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2.
\end{gathered}
\tag{2.7}
$$

In order to determine the sign in the last formula, it is necessary to consider dominating deformation. Thus, for instance, in the case of dominating flexural strain we have,

$$
-2\frac{h^2}{4}P_\varkappa < hP_{\varepsilon\varkappa} < 2\frac{h^2}{4}P_\varkappa.
$$

It is easy to check that this inequality will take place, if in formula (2.7) for $P_\varkappa$ we take sign (+) in parentheses. Analogously we will be convinced that in the case, when dominate stretching — compression deformations of the middle surface predominates one of the inequalities,

$$
z_0 = \frac{P_{\varepsilon\varkappa}}{P_\varkappa} > \frac{h}{2}, \quad z_0 = \frac{P_{\varepsilon\varkappa}}{P_\varkappa} < -\frac{h}{2}.
$$

takes place. This inequality will be fulfilled if for $P_\varkappa$ in parentheses (2.7) we take sign (-).

Thus, subsequently, in all formulas, having two sigrs, the upper sign will pertain to the case of the dominating bend of the shell, and the lower sign to the case of its dominating extension — compression.

We introduce two basic parameters $\lambda$ and $\mu$ in the following manner:

$$
\lambda = \frac{e_{i2}}{e_{i1}}, \quad \mu = \frac{e_{i0}}{e_{i1}}. \tag{2.8}
$$

These parameters satisfy conditions

$$
0 < \lambda > \mu < 1, \tag{2.9}
$$

inasmuch as $e_{i0}$ is the minimum value of intensity of deformations in

a given point of the shell. Then formulas (2.7) can be rewritten in the form:

$$P_{\varepsilon\varkappa} = \frac{3e_{i1}^2}{8k}\Delta\Delta_1,$$
$$P_\varepsilon = \frac{3e_{i1}^2}{16}(4\mu^2+\Delta^2), \qquad (2.10)$$
$$P_\varkappa = \frac{3e_{i1}^2}{4k^2}\Delta_1^2,$$

where $\Delta_1$ and $\Delta$ designate the following functions:

$$\Delta_1 = \left|\sqrt{1-\mu^2} \pm \sqrt{\lambda^2-\mu^2}\right|, \quad \Delta = \frac{1-\lambda^2}{\Delta_1}. \qquad (2.11)$$

The form of formula (2.10) for $P_\varepsilon$ will become quite intelligible, if we were to consider the identity

$$4\mu^2 + \Delta^2 = 1 + \lambda^2 + 2\mu^2 \mp 2\sqrt{(1-\mu^2)(\lambda^2-\mu^2)}.$$

Using designations $\lambda$, $\mu$ and the fixed rule of application of two-digit formulas, we can copy the expression of functions A, B, C in the form:

$$A = \sigma_s e_{i1}\varphi(\lambda, \mu),$$
$$B = \sigma_s\psi(\lambda, \mu), \qquad (2.12)$$
$$C = \frac{1}{2}\sigma_s e_{i1}^2[\chi(\lambda, \mu) - \mu^2\psi(\lambda, \mu)],$$

where functions $\varphi$, $\psi$, and $\chi$ are determined in this manner:

$$\varphi = \lambda - 1,$$
$$\psi = \left|\ln\frac{1+\sqrt{1-\mu^2}}{\mu} \pm \ln\frac{\lambda+\sqrt{\lambda^2-\mu^2}}{\mu}\right|, \qquad (2.13)$$
$$\chi = \left|\sqrt{1-\mu^2} \pm \lambda\sqrt{\lambda^2-\mu^2}\right|.$$

From formulas (2.10) and (2.12) we can now see that quadratic $P_s$, $P_l$, $P_{sl}$ are functions of parameters $\lambda$, $\mu$ only and do not depend on values $e_{i1}$

$$P_s = l_1^2 P_\varepsilon - 2l_1 l_2 P_{\varepsilon\varkappa} + l_2^2 P_\varkappa,$$
$$P_l = l_2^2 P_\varepsilon - 2l_2 l_3 P_{\varepsilon\varkappa} + l_3^2 P_\varkappa, \qquad (2.14)$$
$$P_{sl} = l_1 l_2 P_\varepsilon - (l_1 l_3 + l_2^2) P_{\varepsilon\varkappa} + l_2 l_3 P_\varkappa.$$

[H = l = lower]

Let us note that relationships (2.14) present three algebraic equations, from which forms $P_\varepsilon$, $P_\varkappa$, $P_{\varepsilon\varkappa}$ can be expressed through $P_s$, $P_1$, $P_{s1}$:

$$P_\varepsilon = f_1(P_s, P_m, P_{sm}),$$
$$P_\varkappa = f_2(P_s, P_m, P_{sm}), \tag{2.15}$$
$$P_{\varepsilon\varkappa} = f_3(P_s, P_m, P_{sm}).$$

Indeed, in the first equality (2.14) components have the common factor $\sigma_s^2 h^2$ but do not depend on $e_i$, since $I_1^2$ is reciprocal to $e_{i1}^2$, and $P_\varepsilon$ is directly proportional to $e_{i1}^2$. Analogously, we will be convinced that in the second equality (2.14) components have the common factor $\sigma_s^2 h^4$, and $e_{i1}$ in them is reduced, whereas in the third equality (2.14) components do not depend on $e_{i1}$ and have the common factor $\sigma_s^2 h^3$.

In connection with this it is natural to introduce designations for the characteristic value of forces $T_1$, $T_2$, $T_{12}$ and moments $M_1$, $M_2$, $M_{12}$:

$$T_s = \sigma_s h, \quad M_s = \frac{\sigma_s h^2}{4}. \tag{2.16}$$

Values $T_s$ and $M_s$ respectively, in problems on zero-moment deformations of shells and problems on purely moment deformations play the same role, as the yield point $\sigma_s$ in the problem on the plane stressed state. Therefore, it is expedient to introduce designations for dimensionless forces and moments:

$$t_1 = \frac{T_1}{T_s}, \quad t_2 = \frac{T_2}{T_s}, \quad t_{12} = \frac{T_{12}}{T_s},$$
$$m_1 = \frac{M_1}{M_s}, \quad m_2 = \frac{M_2}{M_s}, \quad m_{12} = \frac{M_{12}}{M_s}, \tag{2.17}$$

and instead of quadratic forms,

$$P_s = \frac{3}{4}(T_1^2 - T_1T_2 + T_2^2 + 3T_{12}^2),$$
$$P_m = \frac{3}{4}(M_1^2 - M_1M_2 + M_2^2 + 3M_{12}^2), \tag{2.18}$$
$$P_{sm} = \frac{3}{4}\left(T_1M_1 + T_2M_2 - \frac{1}{2}T_1M_2 - \frac{1}{2}T_2M_1 + 3T_{12}M_{12}\right)$$

to consider quadratic forms from dimensionless forces and moments,

$$Q_t = t_1^2 - t_1 t_2 + t_2^2 + 3t_{12}^2,$$
$$Q_m = m_1^2 - m_1 m_2 + m_2^2 + 3m_{12}^2, \tag{2.19}$$
$$Q_{tm} = t_1 m_1 - \frac{1}{2} t_1 m_2 - \frac{1}{2} t_2 m_1 + t_2 m_2 + 3t_{12} m_{12}.$$

The latter are connected with $P_s$, $P_1$, $P_{s1}$ (2.18) by evident relationships,

$$Q_t = \frac{4P_s}{3T_s^2}, \quad Q_m = \frac{4P_u}{3M_s^2}, \quad Q_{tm} = \frac{4P_{su}}{3T_s M_s}. \tag{2.20}$$

Performing transformations of right sides of equations (2.14), namely, squaring the polynomials and multiplying, and thereupon assembling coefficients of $\varphi^2$, $\psi^2$, $\varphi\psi$, $\chi\psi$, $\varphi\chi$, $\chi^2$, we obtain the following equations:

$$Q_t = \frac{1}{\Delta_1^2}(\mu^2\psi^2 + \varphi^2),$$
$$Q_{tm} = \frac{2}{\Delta_1^3}(\mu^2\Delta\psi^2 + \Delta\varphi^2 + \mu\varphi\psi + \varphi\chi), \tag{2.21}$$
$$Q_m = \frac{4}{\Delta_1^4}[\mu^2(\mu^2 + \Delta^2)\psi^2 + (4\mu^2 + \Delta^2)\varphi^2 +$$
$$+ 2\mu^2\Delta\varphi\psi - 2\mu^2\psi\chi + 2\Delta\varphi\chi + \chi^2].$$

Inasmuch as the right sides of equations (2.21), according to (2.1 , and (2.13), are functions of only two parameters $\lambda$ and $\mu$, then in a three-dimensional space with variables $Q_t$, $Q_m$, $Q_{tm}$ they represent the surface

$$F(Q_t, Q_m, Q_{tm}) = 0, \tag{2.22}$$

and (2.21) is the parametric equation of this surface. The connection thus obtained, between the quadratic forms (2.19) is called the final relationship between forces and moments, effective in a shell. This fundamental result was obtained by Il'yushin on the basis of Von Mises hypothesis $\sigma_i = \sigma_s$ and therefore, is a generalization of Von Mises

condition, we will also note that the final relationship will have the same form according to St. Venant-Coulomb flow theory also.

We will mention three particular cases of final relationship.

1. The zero-moment stressed state occurs when $\varkappa_1 = \varkappa_2 = \varkappa_{12} = 0$, and here $P_{\varepsilon\varkappa} = 0$ also. Final relationship will be obtained from (2.21), if we assume that deformations of fibers are identical throughout the thickness of the shell,

$$e_{i1} = e_{i2} = e_{i0}, \qquad \lambda = \mu = 1.$$

In formulas (2.11) and (2.13) it is possible to take the lower sign and then to open indeterminite forms in formulas (2.21). Then we find Von Mises condition:

$$Q_m = Q_{tm} = 0, \quad Q_t = 1,$$

or in the expanded form:

$$T_1^2 - T_1T_2 + T_2^2 + 3T_{12}^2 = T_s^2. \tag{2.23}$$

2. The purely moment stressed state takes place in the absence of elongation of the middle surface. Quadratic form $P_\varepsilon = 0$, and therefore, $P_{\varepsilon\varkappa} = 0$. As it follows from formula (1.3), the intensity of deformations $e_i$ is the even function z, and according to (1.12) we have:

$$e_{i1} = e_{i2}, \quad e_{i0} = 0, \quad \lambda = 1, \quad \mu = 0.$$

In formulas (2.11), (2.13) we should take the lower sign, since $z_0 = \frac{P_{\varepsilon\varkappa}}{P_\varkappa} = 0$, thus, we obtain,

$$\Delta_1 = 2, \ \Delta = 0, \ \varphi = 0, \ \psi = 2\ln 2, \ \chi = 2.$$

Final relationship (2.21) takes the form,

$$Q_t = Q_{tm} = 0, \quad Q_m = 1,$$

or

$$M_1^2 - M_1M_2 + M_2^2 + 3M_{12}^2 = M_s^2. \tag{2.24}$$

3. The simplest complex stressed state of shells when $P_\varkappa \neq 0$, $P_\varepsilon \neq 0$, takes place, if the bilinear form ($P_{\varepsilon\varkappa} = 0$) turns into zero,

$$P_{e1} = x_1\left(e_1 + \frac{1}{2}e_2\right) + x_2\left(e_2 + \frac{1}{2}e_1\right) + x_{12}e_{12} = 0. \tag{2.25}$$

It can take place, for instance, in cases,

(a) $x_{12} = x_2 = 0, \quad x_1 \neq 0, \quad e_1 = -\frac{1}{2}e_2;$

(b) $e_{12} = e_2 = 0, \quad e_1 \neq 0, \quad x_1 = -\frac{1}{2}x_2$

and many others.

From (2.7) we now have,

$$e_{t1} = e_{t2} > e_{t0}, \quad \lambda = 1, \quad \mu < 1,$$

i.e., we have the dominating bending strain. We find,

$$\Delta = \varphi = 0, \quad \Delta_1 = \chi = 2\sqrt{1-\mu^2}, \quad \psi = 2\ln\frac{1+\sqrt{1-\mu^2}}{\mu},$$

and after simple transformations the final relationship takes the form,

$$\begin{gathered} Q_t = \frac{\mu^2}{1-\mu^2}\ln\frac{1+\sqrt{1-\mu^2}}{\mu}, \\ Q_{tm} = 0, \\ Q_m = \left(\frac{\mu}{1-\mu^2}\ln\frac{1+\sqrt{1-\mu^2}}{\mu} - \frac{1}{\sqrt{1-\mu^2}}\right)^2. \end{gathered} \tag{2.26}$$

It gives the line of intersection of surface (2.22) with plane $Q_{tm} = 0$. Inasmuch as $Q_t$, $Q_m$ are essentially positive, the entire surface is located between planes $Q_t = 0$ and $Q_m = 0$, and line (2.26) — between positive directions of $Q_t$, $Q_m$ axes, i.e., in the first quadrant of plane $Q_{tm} = 0$. Point $Q_m = 0$, $Q_t = 1$, corresponding to the zero-moment state of the shell, is obtained from (2.26) when $\mu = 1$, and point $Q_t = 0$, $Q_m = 1$, corresponding to the purely moment state of shell, is obtained when $\mu = 0$. The latter is obvious, inasmuch as $\mu \ln \mu = 0$ when $\mu = 0$.

## § 3. Setting Up the Problem of Determination of the Supporting Power.

Having Il'yushin's final relationship it is possible to give the general formulation of the problem of determination of the supporting power of shells.

Indeed, if we assume that forces and moments or quadratic forms $Q_t$, $Q_m$, $Q_{tm}$ are given and satisfy the final relationship (2.22), then with them any two equations (2.22) enable us to find parameters:

$$\lambda = \frac{e_{i2}}{e_{i1}}, \quad \mu = \frac{e_{i3}}{e_{i1}},$$

and then, according to (2.10) and (2.8), to find $I_1$, $I_2$, $I_3$. In this the value $e_{i1}$ will remain indefinite, and we will obtain,

$$I_1 = \frac{1}{e_{i1}} F_1(Q_t, Q_m, Q_{tm}), \quad I_2 = \frac{1}{e_{i1}} F_2(Q_t, Q_m, Q_{tm}),$$
$$I_3 = \frac{1}{e_{i1}} F_3(Q_t, Q_m, Q_{tm}), \tag{3.1}$$

where $F_n$ will be fully definite functions of force and moments.

If these values of integrals $I_1$, $I_2$, $I_3$ are introduced in formulas of forces and moments, then one of the six equations thus obtained will be the result of five others, since forces T and moments M satisfy the final relationship (2.22). Solving these six equations with respect to six deformations and distortions, we will obtain, taking into account (2.27), relationships

$$e_1 = e_{i1}\frac{S_1F_3 - H_1F_2}{\Delta'}, \quad \varkappa_1 = e_{i1}\frac{S_1F_2 - H_1F_1}{\Delta'},$$
$$e_2 = e_{i1}\frac{S_2F_3 - H_2F_2}{\Delta'}, \quad \varkappa_2 = e_{i1}\frac{S_2F_2 - H_2F_1}{\Delta'}, \tag{3.2}$$
$$e_{12} = e_{i1}\frac{S_{12}F_3 - H_{12}F_2}{\Delta'}, \quad \varkappa_{12} = e_{i1}\frac{S_{12}F_2 - H_{12}F_1}{\Delta'},$$

where

$$\Delta' = F_1F_3 - F_2^2,$$
$$S_1 = T_1 - \frac{1}{2}T_2, \quad S_2 = T_2 - \frac{1}{2}T_1, \quad S_{12} = \frac{3}{2}T_{12},$$
$$H_1 = M_1 - \frac{1}{2}M_2, \quad H_2 = M_2 - \frac{1}{2}M_1, \quad H_{12} = \frac{3}{2}M_{12};$$

where in (3.2), if $e_{11}$ has the value (1.12), one of equations is the result of five others; we can easily verify this, if we set up from (3.2) the corresponding quadratic forms.

Inasmuch as six components of deformations and distortions are expressed by means of differential operations on curvilinear coordinates through three components of the vector of displacement w at the point of the middle surface, they have to satisfy deformation compatibility equations. In a general case, the compatibility equation can be expressed only through forces T and moments M, but in the case (3.2) they will contain one more function of coordinates $e_{11}$. Thus, differential equations of equilibrium and conditions of compatibility of deformations will be insufficient for the determination of forces $T_1$, $T_2$, $T_{12}$, moments $M_1$, $M_2$, $M_{12}$ and unknown function $e_{11}$. The needed equation will be the final relationship (2.21) between forces and moments. In view of the fact that this relationship is not differential and from it, it follows that forces and moments and even their quadratic forms $Q_t$, $Q_m$, $Q_{tm}$ are limited in value, it is clear that with arbitrary external forces the equilibrium of the shell is impossible.

The supporting power of a shell is a term given to that limiting value of external forces, with which internal forces T and moments M satisfy the final relationship (2.21), equations of equilibrium, conditions of compatibility of deformations and boundary conditions. In certain particular cases owing to the final relationship the problem of equilibrium becomes statically determinable and does not require conditions of compatibility of deformations. Then the problem of the supporting power of the shell is resolved comparatively simply. It is simplified even more, if forces and moments can be expressed

through external forces only by means of equations of equilibrium which takes place, for instance, in the zero-moment theory of shells; in such case, the final relationship (2.21) determines the supporting power.

Conditions of compatibility of deformations render the problem of determination of the supporting power very complicated and therefore, approximation methods of its solution are of great importance. The energy method of solution consists of the following: we prescribe a suitable form of deformed surface of shells and, setting up expressions of variations of the work of internal forces and work of external forces on variations of displacements, compare them. The approximate limiting value of external forces will be obtained either if hardening of the material is assumed to be equal to zero, but deformations are increased without a limit, or which is the same, preserving the constant the yield point $\sigma_s = 3Ge_s$, G is made to approach infinity, and $e_s$ — to approach zero.

## § 4. Determination of the Supporting Power and Work-Hardening of Shells.

Let us examine following Il'yushin's relationship [2], the supporting power and work-hardening* of shells. We present external forces in the form of internal pressure p and the resultant force P, which stretches the shell in the direction of 00 axis, (Fig. 69); force P is a projection on the external forces of the axis which act on the part of the shell located either on the right or on the left of the section. Since the stressed state of shell is zero-moment, meridional stretching force $T_1$ and tangential force $T_2$ are connected

*See below, p. 405

with corresponding stresses $\sigma_1$, and $\sigma_2$ with simple formulas:

$$T_1 = \sigma_1 h, \quad T_2 = \sigma_2 h.$$

Equations of equilibrium of vessels are known and have the form:

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} = p, \quad 2\pi r T_1 \sin\theta = P, \tag{4.1}$$

and stresses $\sigma_1$ and $\sigma_2$ are determined by formulas:

$$\begin{gathered} \sigma_1 = \frac{rp}{h} s_1, \quad \sigma_2 = \frac{rp}{h} s_2, \\ s_1 = \frac{PR_2}{2\pi r^2 p}, \quad s_2 = \frac{R_2}{r} - \frac{R_2}{R_1} s_1. \end{gathered} \tag{4.2}$$

The intensity of stresses is determined by the formula

$$\sigma_i = |p| \frac{r}{h} \sqrt{s_1^2 - s_1 s_2 + s_2^2}. \tag{4.3}$$

Thus, in the case of small deformations of shell, for the total determination of the stressed state it is sufficient to have equations of statics only.

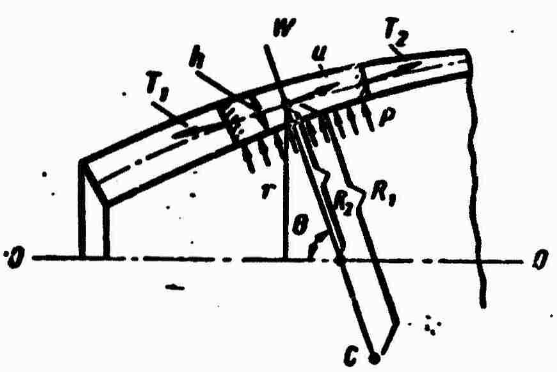


Fig. 69.

If the shell material does not have work-hardening, then from Von Mises' condition

$$\sigma_i = \sigma_s$$

we find the critical load, which the shell, will withstand, i.e., the supporting power of shell;

$$|p| \sqrt{s_1^2 - s_1 s_2 + s_2^2} = \frac{h}{r} \sigma_s. \tag{4.4}$$

If the load acting in the section is less than the critical load, i.e., the left part of equality (4.4) is less than the right part, then deformation of the shell in this section is elastic; otherwise, equilibrium of forces is impossible.

For shells, the material of which has been work-hardened and is characterized by the diagram $\sigma_i = \Phi(e_i)$

$$(\text{or}\quad \sigma_i = Ee_i(1-\omega),\quad Ee_i = \sigma_i(1+\varphi)),$$

so that with diagram $\sigma_i \sim e_i$ function $\varphi(\sigma_i)$ is determined according to given forces p, P or stresses $\sigma_1$, $\sigma_2$ we can easily find the meridional and tangential components of the deformation of the middle surface:

$$\begin{aligned}\varepsilon_1 &= \frac{e_i}{\sigma_i}\left(\sigma_1 - \frac{1}{2}\sigma_2\right),\\ \varepsilon_2 &= \frac{e_i}{\sigma_i}\left(\sigma_2 - \frac{1}{2}\sigma_1\right),\end{aligned} \tag{4.5}$$

or according to (4.2):

$$\begin{aligned}\varepsilon_1 &= \frac{(1+\varphi)rp}{Eh}\left(s_1 - \frac{1}{2}s_2\right),\\ \varepsilon_2 &= \frac{(1+\varphi)rp}{Eh}\left(s_2 - \frac{1}{2}s_1\right).\end{aligned} \tag{4.6}$$

Components of the vector of displacement of the point of the section considered in the directions of the external normal and generator toward the growth of angle $\theta$ we designate with w and u, respectively (Fig. 69). Then formulas, expressing deformations through displacements, will take the form:

$$\begin{aligned}\varepsilon_1 &= \frac{w}{R_1} + \frac{du}{R_1 d\theta},\\ \varepsilon_2 &= \frac{w\sin\theta + u\cos\theta}{r}.\end{aligned} \tag{4.7}$$

We consider the displacement equal to zero in the section, where $\theta = \frac{\pi}{2}$, i.e., where the generator of the shell is parallel to its axis. In such case, integrating the differential equation for displacement u, which is obtained from (4.7) by means of exception of w, we find:

$$u = \sin\theta \int_{\frac{\pi}{2}}^{\theta} \frac{\varepsilon_1 R_1 - \varepsilon_2 R_2}{\sin\theta} d\theta,$$

$$w = \varepsilon_2 R_2 - u\,\mathrm{ctg}\theta.$$

Introducing here values $\varepsilon_1$ and $\varepsilon_2$, we obtain the final expressions of displacements;

$$
\begin{aligned}
u &= \frac{\sin\theta}{E}\int_{\frac{\pi}{2}}^{\theta}\frac{pr(1+\varphi)}{h\sin^2\theta}\left[s_1\left(R_1+\frac{1}{2}R_2\right)-s_2\left(R_2+\frac{1}{2}R_1\right)\right]d\theta,\\
w &= \frac{r(1+\varphi)p}{Eh}\left(s_2-\frac{1}{2}s_1\right)R_2-u\,\mathrm{ctg}\,\theta.
\end{aligned}
\tag{4.8}
$$

Here function $\varphi$ is assumed to be expressed through the intensity of stresses $\sigma_i$, which , in turn, is determined through known values by the formula (4.3). Inasmuch as the elastic displacements (in the case, when $\sigma_i < \sigma_s$) are obtained by formulas (4.8), in which it is necessary to assume $\varphi = 0$, then it is clear that residual displacements $\tilde{u}$ and $\tilde{w}$, which are retained after removal of the load, are also obtained from formulas (4.8), if in them instead of $(1 + \varphi)$ we retain only the value $\varphi$.

For shells, the material of which has linear strengthening, function $\varphi(\sigma_i)$ has the expression:

$$
\varphi = 0,\quad \sigma_i < \sigma_s,\quad \varphi = \frac{\lambda}{1-\lambda}\left(1-\frac{\sigma_s}{\sigma_i}\right),\quad \sigma_i > \sigma_s,\quad \lambda = 1-\frac{1}{E}\frac{d\sigma_i}{de_i}. \tag{4.9}
$$

If we use this expression of $\varphi$ and, replacing in it $\sigma_i$ according to formula (4.3), substitute it in (4.8), we can obtain general and residual deformations of shells in a clear form. However, the integral included in (4.9) can be calculated only after we are given the form and dimensions of the shell, as well as the load.

If the shell material has a significant strengthening, so that, for instance, the true resistance in the break of sample is twice as large as the yield point, then by means of work-hardening,* we can significantly increase the durability of the shell.

---

*Work-hardening is the term given to the process of hardening of the shell by giving it a preliminary plastic deformation of a comparatively large value.

In the case of a spherical shell, the initial radius of which is $R_0$, the thickness of wall $h_0$, and final dimensions are R and h respectively, we have,

$$\sigma_1 = \sigma_2 = \sigma_i = \frac{R}{2h} p,$$

here p is the final value of internal pressure. Deformations $\varepsilon_1$ and $\varepsilon_2$ are identical and are determined by the formula

$$\varepsilon_1 = \varepsilon_2 = \frac{R - R_0}{R_0} = \rho - 1, \quad \rho = \frac{R}{R_0}. \tag{4.10}$$

From formulas (4.5) we have,

$$\rho - 1 = \frac{1+\varphi}{E}\left(\sigma_1 - \frac{1}{2}\sigma_2\right) = \frac{1+\varphi}{2E}\sigma_i. \tag{4.11}$$

From the condition of permanency of the shell's mass

$$4\pi R^2 h = 4\pi R_0^2 h_0, \tag{4.12}$$

and therefore, the expression of intensity of stresses $\sigma_i$ can be transformed to form

$$\sigma_i = \frac{R_0 p}{2h_0} \rho^3. \tag{4.13}$$

Inasmuch as the characteristic of hardening of material $\varphi(\sigma_i)$ is known, then equation (4.11) determines pressure p, which, the greatly deformed spherical shell, can withstand.

Let us use the law of linear hardening (4.9) and determine this pressure,

$$p = \lambda \frac{1 + m(\rho - 1)}{\rho^3} p_s, \tag{4.14}$$

where $p_s$ is the pressure, at which yield of the blank begins, and m is the parameter, depending on the elongation $e_s = \frac{\sigma_s}{E}$ and on $\lambda$:

$$p_s = \frac{2h_0}{R_0}\sigma_s, \quad m = \frac{2(1-\lambda)}{\lambda e_s}. \tag{4.15}$$

Determining the maximum p according to ρ, we find that the corresponding value of deformation

$$\rho_m = \frac{R_m}{R_0}$$

will be

$$\rho_m = \frac{3}{2}\frac{m-1}{m},$$

and the maximum resistance of the strengthened shell is equal to:

$$p_m = \frac{4m^3}{27(m-1)^2} p_s. \qquad (4.16)$$

Further work-hardening is inexpedient because it is accompanied by thinning of the wall, which weakens the shell more than it is strengthened through cold hardening of the material. Let us note that the maximum strength given by formula (4.16) is not always attainable for metal shells, since deformation $\rho_m$ can be larger than the deformation with which a break occurs. But formula (4.14) shows that work-hardening, even if it is insignificant, increases durability very effectively. For instance, for steel, having $\lambda = 0.98$, $e_s = 2 \times \times 10^{-3}$, we have $m \approx 20$, the inflation of spherical shell by 5% only ($\rho = 1.05$) gives $p = 1.7p_s$, i.e., increases its elastic limit by 70%.

In the case of the cylindrical shell, which is deformed by uniform internal pressure, so that $\sigma_1 = \frac{1}{2}\sigma_2$, corresponding formulas have the form,

$$p = \lambda \frac{1 + m(\rho - 1)}{\rho^2} p_s, \quad p_m = \frac{h_0}{R_0\sqrt{3}}\sigma_s, \quad m = \frac{2(1-\lambda)}{\lambda e_s \sqrt{3}}, \qquad (4.17)$$

where the most advantageous work-hardening is determined by formulas.

$$\rho_m = 2\frac{m-1}{m}, \quad p_m = \lambda \frac{m^2}{4(m-1)} p_s.$$

## § 5. Energy Method of Determination of the Supporting Power

For determination of the supporting power of shells (and plates) we can successfully use the kinematic principle of determination of the breaking load, which was formulated and proven for the first time by A. A. Gvozdev [77, 78] and developed by A. R. Rzhanitsyn [79].

This principle consists of the fact that from all possible forms of destruction of an elastoplastic system the effective form of destruction will be that form, for which the value of the given external load, balanced on this form of destruction, will be the least. As the possible forms of destruction we usually investigate mechanisms, in which the given system rotates when in it a certain quantity of elastic bonds are replaced by plastic bonds, which can be deformed in a definite direction without an increase of the attained critical binding force. However, as A. R. Rzhanitsyn [79] notes, it is possible to proceed from a more general system of destruction, including all mechanisms and constituting a kinematic chain, displacements of which are determined, let us say, by k parameters. Making k - 1 of these of the mechanisms which correspond to the destructive state of the system.

Thus, the breaking load can be approached by investigating all mechanisms, which are obtained from a certain common kinematic chain formed from the prescribed system by replacing in it elastic bonds by plastic ones by means of equating to zero of all but one displacement parameters of the kinematic chain. Generalizing this method, we can instead of equating to zero k - 1 displacement parameters of the kinematic chain introduce k - 1 relationships between these parameters.

Let us assign the body under examination the greatest possible number of degrees of freedom, granting the existence of flow

deformation in any of its points. We will also consider volume deformations in all points of the body to be equal to zero or to be so small as compared to plastic deformations of the change of shape that their work can be disregarded. For the condition of plasticity, we will adopt that formulated by Von Mises:

$$\sigma_i = \sigma_s = \text{const.} \tag{5.1}$$

Unit work of internal stresses under a simple load according to [2] is equal to

$$dT = \left[\int^{e_i} \sigma_i de_i + K\frac{\theta^2}{2}\right] dx\,dy\,dz. \tag{5.2}$$

Here $\theta$ is the volume deformation, K is the volume of elastic modulus.

The second right-hand term (5.2), which depends on volume deformation, according to what was said above, can be disregarded.

Assuming that the material is not strengthened, from (5.2) we obtain:

$$dT = \sigma_s e_i\,dx\,dy\,dz. \tag{5.3}$$

Total work of internal forces is equal to

$$T = \sigma_s \iiint e_i\,dx\,dy\,dz, \tag{5.4}$$

where integration is performed over the entire volume of the body.

The work of external forces is determined, as usually, by formula

$$A=\iiint(Xu+Yv+Zw)\,dx\,dy\,dz, \tag{5.5}$$

where X, Y, and Z are volume forces; u, v, w are displacements of points of the body. To the volume forces we must add also the surface forces, which we can consider included in the same integral (5.5), considering the surface forces to be classified as infinitesimal volumes, located immediately next to the surface of the body.

The work of internal forces T can be expressed through transpositions of u, v, w, if we use the well-known Cauchy formulas; substituting these formulas in the expression of intensity of deformation, and then in (5.4), we will obtain.

$$\begin{aligned}T=\frac{\sqrt{2}}{3}\sigma_s\iiint\Big\{2\Big[\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial v}{\partial y}\Big)^2+\Big(\frac{\partial w}{\partial z}\Big)^2-\\-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}\Big]+\frac{3}{2}\Big[\Big(\frac{\partial u}{\partial y}\Big)^2+\\+\Big(\frac{\partial u}{\partial z}\Big)^2+\Big(\frac{\partial v}{\partial x}\Big)^2+\Big(\frac{\partial v}{\partial z}\Big)^2+\Big(\frac{\partial w}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial y}\Big)^2\Big]+\\+3\Big[\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}\Big]\Big\}^{\frac{1}{2}}dx\,dy\,dz.\end{aligned} \tag{5.6}$$

The problem is now reduced to finding such functions

$$u=u(x,y,z),\quad v=v(x,y,z),\quad w=w(x,y,z),$$

with which expression (5.6), or, which is the same (5.4), would assume the minimum value under the additional condition:

$$A=\iiint(Xu+Yv+Zw)\,dx\,dy\,dz=\mathrm{const} \tag{5.7}$$

and the condition of the absence of volume deformations:

$$\varepsilon_1+\varepsilon_2+\varepsilon_3=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{5.8}$$

Solving this problem by the usual methods of variation calculus, after a number of transformations we will obtain a complicated system of three nonlinear partial differential equations, the solution of

which is very difficult.

However, it is possible to apply here the approximation method of solution of variational problems, in which we must prescribe functions u, v, w approximately with the precision up to a small number of parameters and to find these parameters from the condition of the minimum of expression (5.6) observing conditions of (5.7). Furthermore, the prescribed functions must satisfy (5.8) the condition of absence of volume deformations.

The number of indefinite parameters of displacement functions in many cases can be reduced to one, if every time we prescribe successfully the form of these functions, for instance, assuming beforehand the picture of distribution of deformations and displacements in a plastic body under a given load.

Using approximate expressions for displacements, we can obtain the value of the limit load from relationship

$$A = T. \tag{5.9}$$

Here the limit load will be always somewhat larger than its real value, obtained from the exact solution, inasmuch as the minimum of T in the approximate solution is not fully attained.

Let us consider A. R. Rzhanitsyn's example on the application of method [80]. Let us assume that a cylindrical shell open on one end is loaded on the free edge by evenly distributed radial forces q (Fig. 70). Let us prescribe a deformation, with which at a certain length c the shell is turned into frustum of a cone, while in it appear annular elongations $\varepsilon_\varphi$, equal to $(c - x)\varphi\frac{1}{R}$, and longitudinal elongations $\varepsilon_x = -\frac{1}{2}\varepsilon_\varphi$. Here x is the coordinate of annular section, read off the free end of shell, R is the radius of section of the shell, $\varphi$ is the angle of inclination of generators in the deformed conical

part. We will designate the thickness of the shell wall with $\delta$. In section $x = c$ during the given deformation, an annular hinge of fluidity and angle $\varphi$ of break of shell generators appear. It is not difficult to calculate the work of internal forces in the given deformation. Annular elongations will yield:

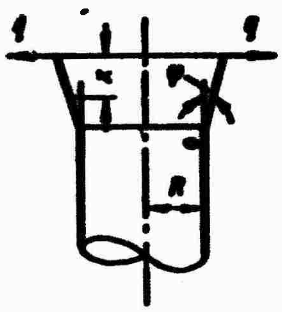

Fig. 70.

$$\sigma_s \delta 2\pi R \int_0^c (c-x)\frac{\varphi}{R}\,dx = \pi\sigma_s c^2\delta\varphi,$$

and the break in annular section $x = c$

$$\frac{2}{\sqrt{3}}\varphi 2\pi R m_s = \frac{\pi R\delta^2\sigma_s\varphi}{\sqrt{3}}.$$

Thus, the work of internal forces will be equal to:

$$T = \pi\sigma_s\varphi\delta\left(c^2+\frac{R\delta}{\sqrt{3}}\right). \tag{5.10}$$

The work of external forces is equal to

$$A = 2\pi R q c\varphi. \tag{5.11}$$

According to (5.9) we obtain:

$$q = \frac{\sigma_s\delta}{2R}\left(c+\frac{R\delta}{c\sqrt{3}}\right). \tag{5.12}$$

q will be at a minimum when:

$$c = \sqrt{\frac{R\delta}{\sqrt{3}}} = 0{,}76\sqrt{R\delta}; \tag{5.13}$$

it is equal to

$$q_{\text{пр}} = \frac{1}{\sqrt[4]{3}}\sigma_s\delta\sqrt{\frac{\delta}{R}} = 0{,}76\,\sigma_s\delta\sqrt{\frac{\delta}{R}}. \tag{5.14}$$

[пр = lim = limit]

If the free edge of shell will be loaded in addition to radial forces q, also with distributed moments m (Fig. 71), then the work of external forces will be equal to

$$A = 2\pi R\varphi(m+qc). \tag{5.15}$$

Here according to (5.9) and from (5.10) and (5.15) it follows that:

$$m + qc = \frac{\sigma_s \delta}{2R}\left(c^2 + \frac{R\delta}{\sqrt{3}}\right). \tag{5.16}$$

For every value of c we obtain a relationship between m and q in the form of a straight line (in right-angle coordinates q, m)

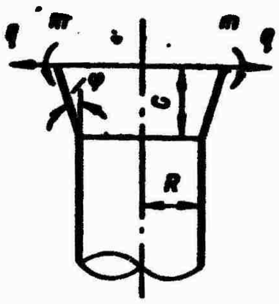


Fig. 71.

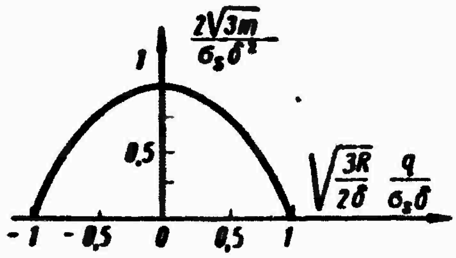


Fig. 72

The envelope of the family of these straight lines will limit the region of limit values of q and m. In order to find the envelope, we differentiate expression (5.16) with respect to c:

$$q = \frac{\sigma_s \delta}{R} c,$$

and, having determined c from here we substitute in (5.16). We will obtain:

$$\begin{aligned} m + \frac{R}{\sigma_s \delta} q^2 &= \frac{\sigma_s \delta}{2R}\left(\frac{R^2 q^2}{\sigma_s^2 \delta^2} + \frac{R\delta}{\sqrt{3}}\right) = \\ &= \frac{Rq^2}{2\sigma_s \delta} + \frac{\sigma_s \delta^2}{2\sqrt{3}}. \\ m + \frac{R}{2\sigma_s \delta} q^2 &= \frac{\sigma_s \delta^2}{2\sqrt{3}} = \frac{2}{\sqrt{3}} m_s. \end{aligned} \tag{5.17}$$

Thus, the region of limit values m and q constitutes a parabola with the summit on the m axis (Fig. 72).

Let us modify the preceding problem, applying radial load q on the average section of a long cylindrical shell, as shown in Fig. 73. In this case we have the following expression for the work of internal forces:

$$T = \pi \sigma_s \varphi \delta \left( 2c^2 + \frac{4}{\sqrt{3}} R\delta \right). \tag{5.18}$$

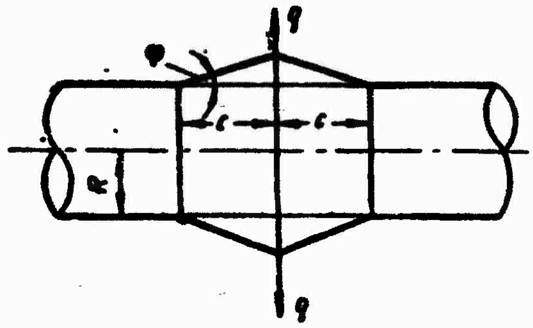

Fig. 73.

The expression for the work of external forces remains the same as before (5.11). Hence

$$q = \frac{\sigma_s \delta}{R} \left( c + \frac{2R\delta}{c\sqrt{3}} \right). \tag{5.19}$$

The minimum value of load is equal to:

$$q_{\text{кр}} = \frac{\sqrt{8}\sigma_s \delta}{\sqrt[4]{3}} \sqrt{\frac{\delta}{R}} = 2{,}14 \sigma_s \delta \sqrt{\frac{\delta}{R}} \tag{5.20}$$

when

$$c = \frac{\sqrt{2}}{\sqrt[4]{3}} \sqrt{R\delta} = 1{,}07 \sqrt{R\delta}.$$

## § 6. Shells With Nonuniform Mechanical Properties

Heterogeneity of mechanical properties throughout the thickness of the shell wall can appear through various causes and can be created artificially for the purpose of strengthening. For instance, at elevated temperatures the yield point $\sigma_s$, weakening coefficient $\lambda$, and elastic moduli G and K can be by variables depending on a significant temperature gradient; in the case of action of a neutron flux heterogeneity of mechanical properties appears also; special heat treatment, resulting in variable distribution of hardness, can be used in a

number of cases to improve operating properties of shell.

V. Olszak [81-85] examined the effect of compressibility, gave a general setting of elastoplastic problems for bodies of arbitrary shape, and solved a number of problems.

The work of the author jointly with A. A. Il'yushin [41], gives a general solution of the problem on deformation of a cylinder with nonuniform mechanical properties, it being elementary, on the assumption of incompressibility of the material.

Here we will examine only the problem of the value of the biggest internal pressure $p_s$ in a cylinder with field of temperature variable with respect to the radius, and a yield point of material $\tau_s(\vartheta)$ which changes depending on the temperature, where pressure $p_s$ is determined so that with any r the deformation of the cylinder remains elastic and only on one circumference $r = r_s$ (or on several simultaneously) the yield point $\tau_m^* = \tau_s$ is attained. The method coincides completely with the method applied by us in work [86] in solving the problem on the influence of irradiation. Elastic moduli are significantly less dependent on the temperature than the yield point and they must be considered constants for the range of temperatures, which is characteristic for the cylinder.

Let us assume the $\vartheta$ is the characteristic constant temperature, which is selected arbitrarily. Let us introduce designations:

$$\tilde{p} = \frac{(1-\nu)}{E\alpha_1\vartheta_0} p, \quad \tilde{\tau}_s = \frac{(1-\nu)}{E\alpha_1\vartheta_0} \tau_s;$$
$$T = \frac{1}{\vartheta_0} \vartheta(\zeta, t), \quad \tilde{\tau} = \frac{(1-\nu)}{E\alpha_1\vartheta_0} \tau_m; \tag{6.1}$$
$$\left(\zeta = \frac{r}{b}\right).$$

Here p is the internal pressure, $\nu$ is Poisson's ratio, $\alpha_1$ is the coefficient of linear temperature expansion, b is the outside radius of cylinder, and a is the inside radius, $\alpha = \frac{a}{b}$. On the basis of solution [87] we have the following expression $\tilde{\tau}$:

$$\tilde{\tau} = \frac{\alpha^2}{1-\alpha^2}\frac{1}{\zeta^2}\left(\tilde{p} + \int_{\alpha}^{1} T\zeta\, d\zeta\right) + \frac{1}{\zeta^2}\int_{\alpha}^{\zeta} T\zeta\, d\zeta - \frac{1}{2}T. \tag{6.2}$$

Yield point $\tilde{\tau}_s$ (6.1) is a known function of temperature $T(\zeta, t)$, which is considered a known function of radius ($\zeta = \frac{r}{b}$) and time t.

According to the condition of the problem when $r = r_s$, $\zeta = \zeta_s = \frac{r_s}{b}$ equality of values $\tilde{\tau}$ (6.2) and $\tilde{\tau}_s(T)$ is attained:

$$\zeta = \zeta_s, \quad \tilde{\tau}_s(T) - \tilde{\tau} = 0,$$

where in the vicinity of $\zeta = \zeta_s$, i.e., when $\zeta \geq \zeta_s$, and $\zeta \leq \zeta_s$, we should have $\tilde{\tau}_s(T) - \tilde{\tau} > 0$. This means, that the full system of conditions when $\zeta = \zeta_s$ has the following form:

$$\begin{gathered} \tilde{\tau}_s(T) - \tilde{\tau} = 0, \\ \frac{\partial}{\partial\zeta}[\tilde{\tau}_s(T) - \tilde{\tau}] = 0, \\ \frac{\partial^2}{\partial\zeta^2}[\tilde{\tau}_s(T) - \tilde{\tau}] > 0, \end{gathered} \tag{6.3}$$

and the system of two equations, determining an unknown circumference $\zeta = \zeta_s$ and the sought highest pressure $\tilde{p} = \tilde{p}_s$, will be:

$$\frac{\alpha^2}{1-\alpha^2}\left(\tilde{p} + \int_{\alpha}^{1} T\zeta\, d\zeta\right) + \int_{\alpha}^{\zeta} T\zeta\, d\zeta - \zeta^2\left(\tilde{\tau}_s + \frac{1}{2}T\right) = 0,$$

$$\frac{\alpha^2}{1-\alpha^2}\left(\tilde{p} + \int_{\alpha}^{1} T\zeta\, d\zeta\right) + \int_{\alpha}^{\zeta} T\zeta\, d\zeta - \frac{1}{2}\zeta^2 T +$$
$$+ \frac{1}{2}\zeta^3\left(\frac{d\tilde{\tau}_s}{dT} + \frac{1}{2}\right)\frac{\partial T}{\partial\zeta} = 0.$$

Hence, subtracting the first relationship from the second, we find (when $\zeta = \zeta_s$):

$$\left(\frac{d\tilde{\tau}_s}{dT}+\frac{1}{2}\right)\frac{\partial T}{\partial \zeta}+\frac{2}{\zeta}\tilde{\tau}_s=0, \tag{6.4}$$

$$\tilde{p}_s=\frac{1-\alpha^2}{\alpha^2}\left[\zeta^2\left(\tilde{\tau}_s+\frac{1}{2}T\right)-\int_\alpha^\zeta T\zeta\,d\zeta\right]-\int_\alpha^1 T\zeta\,d\zeta. \tag{6.5}$$

Equation (6.4) determines radius $r_s(\zeta_s)$; pressure $p_s$ is determined by equation (6.5).

The solution adduced loses its meaning, if inside the region $1 > \zeta > \alpha$ there is no minimum $\tilde{\tau}_s - \tilde{\tau}$ and this difference is monotonously changing throughout the section. Then its least value must be equated to zero, and from this we will find the value of force $\tilde{p}_s$. For instance, if this difference increases from the inner to the outer surface, then $\tilde{p}_s$ is found, when $\zeta = \alpha$, from condition $\tilde{\tau}_{sa} - \tilde{\tau}_{\zeta=\alpha}=0$:

$$\tilde{p}_s=(1-\alpha^2)\left(\tilde{\tau}_{sa}-\frac{1}{2}T_a\right)-\int_\alpha^1 T\zeta\,d\zeta. \tag{6.6}$$

If the temperature field of cylinder is radial and stationary, then the distribution of temperatures will be

$$T=T_b+\frac{T_a-T_b}{\ln\alpha}\ln\zeta. \tag{6.7}$$

Formula (6.4) for $\zeta_s$ assumes the form

$$\frac{d\tilde{\tau}_s}{dT}+\frac{2\tilde{\tau}_s\ln\alpha}{T_a-T_b}+\frac{1}{2}=0. \qquad (\zeta=\zeta_s). \tag{6.8}$$

Hence $\zeta_s$ is found with the aid of curves of dependency of $\tau_s$ on T, with a T prescribed with respect to $\zeta$ (6.7); thereupon $\tilde{p}_s$ is found by (6.5).

Formula (6.6) assumes the form

$$\tilde{p}_s=(1-\alpha^2)\tilde{\tau}_{sa}+\frac{1}{2}(T_a-T_b)\,\frac{1-\alpha^2+\ln\alpha^2}{\ln\alpha^2}. \tag{6.9}$$

## § 7. Shells Subjected to Heating by Radiation

The problem of thermoelasticity for shells (and plates) in the presence of radiation is of practical interest. We know the method of setting-up and solving this kind of problem, devised by V. V. Bolotin [88]. Let us present the consideration for formulation of the problem on deformations of plates and shells, subjected to sudden heating by radiation [89]. Solution of these problems requires an examination of heat-conduction equations and equations of thermoelasticity. Taking into account that in setting-up equations of thermoelasticity in the theory of plates and shells essential use is made of the Kirchhoff-Love hypothesis on the preservation of the normal element, it is justifiable to introduce a similar hypothesis in heat-conduction equations also. Let us assume that temperature T, measured from a certain constant level, is determined by expression

$$T = T_0(x_1, x_2) + x_3\theta(x_1, x_2), \tag{7.1}$$

where $x_1$, $x_2$ are curvilinear coordinates of the middle surface, $x_3$ is the coordinate measured off on the normal to the middle surface. In order to work out the equation for the average temperature $T_0$ and temperature gradient $\Theta$, we will use the variation principle $\delta I = 0$ for functional

$$I = \int_{t_0}^{t_1} \left\{ \frac{1}{2} \int_v \left[ c\rho \left( T^* \frac{\partial T}{\partial t} - T \frac{\partial T^*}{\partial t} \right) + \lambda \nabla_i T \nabla^i T^* - q(T - T^*) \right] dv + \right.$$
$$\left. + \int_s k(TT^* - T_n T - T_n T^*) ds \right\} dt. \quad [\text{H} = \text{n}] \tag{7.2}$$

Here c is the specific heat; ρ is the denisty of material; λ is the coefficient of thermal conduction; q is the density of thermal sources, equal to the amount of heat, produced in a unit of volume of time; k is the heat transfer coefficient of the shell's surface; $T_n$ is the ambient temperature; v is the volume of the shell; s is its surface;

$t_0$ and $t_1$ are two arbitrarily selected moments of time; $\nabla_i$ and $\nabla^i$ are symbols of covariant and contravariant differentiation. With $T^*$ we designate the temperature of the process, proceeding in the sense, opposite to the process considered. It is easy to verify that Ostrogradskiy-Euler's variational equations and natural boundary conditions for functional (7.2) coincide with the heat-conduction equation and condition of convection heat exchange on surface s.

Introducing expression (7.1) in (7.2), we will replace integration with respect to volume v by integration with respect to the middle surface, and integration with respect to surface s by integration with respect to external and internal surfaces of the shell, and also with respect to its ends. Besides we will consider that the thickness of the shell is minute as compared to the radii of curvature of the middle surface. The Ostrogradskiy-Euler equation for the transformed functional assumes the form:

$$\frac{\partial T_0}{\partial t} - \chi\nabla^2 T_0 + \frac{1}{c\rho h}\left[(k_+ + k_-)T_0 + (k_+ - k_-)h\theta\right] = \frac{1}{c\rho h}(Q + k_+T_+ + k_-T_-), \tag{7.3}$$

$$\frac{\partial \theta}{\partial t} - \chi\nabla^2\theta + \frac{12\chi}{h^2}\theta + \frac{6}{c\rho h^2}\left[(k_+ - k_-)T_0 + \frac{k_+ - k_-}{2}h\theta\right] = \frac{6}{c\rho h^2}\left(\frac{2Qz_0}{h} + k_+T_+ - k_-T_-\right), \tag{7.4}$$

where $\nabla^2 = \nabla_i\nabla^i$ is the Laplacian operator, $\chi = \frac{\lambda}{c\rho}$ is the coefficient of thermal conductivity; $T_+$ and $T_-$ are the temperature of medium inside and outside of the shell; $k_+$ and $k_-$ are the corresponding coefficients of heat emission; Q is the density of thermal sources per unit of area of the middle surface; $z_0$ is the coordinate of "the center of gravity" of sources. Natural boundary conditions lead to the following conditions of heat exchange on the ends:

$$\lambda\nabla^i T_0 n_i + k_\gamma (T_0 - T_\gamma) = 0, \tag{7.5}$$

$$\lambda\nabla^i \theta n_i + k_\gamma (\theta - \theta_\gamma) = 0. \tag{7.6}$$

Here $n_i$ is the vector of the normal to end surface and $k_\gamma$ is the coefficient of heat emission for this surface.

Let us examine example [89]. Let us assume that an unlimited plate of constant thickness h, resting at $t < 0$, is subjected in moment of time $t = 0$ to the action of radiation, the intensity of which later drops in time. The field of radiation is assumed to be axisymmetric. Let us assume that between the surface of plate and the medium the conditions of convection heat exchange with heat emission coefficient k are realized, and temperature of medium $T_+ = T_- = 0$. Equations (7.3) and (7.4) assume the following form:

$$\frac{\partial T_0}{\partial t} - \chi\nabla^2 T_0 + \frac{2kT_0}{cph} = \frac{Q}{cph}, \tag{7.7}$$

$$\frac{\partial\theta}{\partial t} - \chi\nabla^2\theta + \left(\frac{12\chi}{h^2} + \frac{6k}{cph}\right)\theta = \frac{12Qz_0}{cph^2}. \tag{7.8}$$

Equations (7.7) and (7.8) must be examined together with equations of thermoelasticity. The equation of two-dimensional axisymmetric problems, written in displacements, has the following form:

$$\nabla^2 u - \frac{u}{r^2} + \frac{\rho(1-\nu^2)}{E}\frac{\partial^2 u}{\partial t^2} = \alpha(1+\nu)\frac{\partial T_0}{\partial r}, \tag{7.9}$$

where u is the radial displacement; $\alpha$ is the coefficient of thermal expansion; $\nu$ and E are Poisson's ratio and elastic modulus. At the same time the equation for the flexure will be

$$\nabla^2\nabla^2 w - \frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} = \alpha(1+\nu)\nabla^2\theta, \tag{7.10}$$

where w is the normal sag; and D is the cylinder rigidity. Equations (7.7)-(7.10) are integrated under initial conditions:

$$T_0(r, 0) = \theta(r, 0) = u(r, 0) = \frac{\partial u(r, 0)}{\partial t} = w(r, 0) = \frac{\partial w(r, 0)}{\partial t} = 0$$

and under boundary conditions, requiring the limitedness of all functions when r = 0 and proper damping of all functions when r tends toward infinity. Here, of course, the density of sources Q(r, t) must be also subjected to the corresponding limitations.

Solution of equations (7.7)-(7.10) is sought for with the help of Hankle's transformation [90]. The temperature field is determined by the formulas:

$$T_0(r, t) = \int_0^\infty T_0^*(p, t) I_0(pr) p\,dp,$$
$$\theta(r, t) = \int_0^\infty \theta^*(p, t) I_0(pr) p\,dp, \tag{7.11}$$

where

$$T_0^*(p, t) = \frac{1}{c\rho h}\int_0^t \exp[-(\chi p^2 + a_0)(t-\tau)] Q^*(p, \tau)\,d\tau,$$
$$\theta^*(p, t) = \frac{12 z_0}{c\rho h^3}\int_0^t \exp[-(\chi p^2 + a)(t-\tau)] Q^*(p, \tau)\,d\tau, \tag{7.12}$$
$$Q^*(p, t) = \int_0^\infty Q(p, t) I_0(pr) r\,dr.$$

While

$$a_0 = \frac{2k}{c\rho h}, \quad a = \frac{12\chi}{h^2} + \frac{6k}{c\rho h}.$$

For the displacement field analogously, we will obtain:

$$u(r, t) = \int_0^\infty u^*(p, t) I_1(pr) p\,dp,$$
$$w(r, t) = \int_0^\infty w^*(p, t) I_0(pr) p\,dp, \tag{7.13}$$

where

$$u^*(p, t) = ag_0^2(1+\nu)\int_0^t \sin[g_0 p(t-\tau)] T_0^*(p, \tau)\,d\tau,$$
$$w^*(p, t) = ag^2(1+\nu)\int_0^t \sin[gp^2(t-\tau)] \theta^*(p, \tau)\,d\tau. \tag{7.14}$$

Here

$$g_0^2 = \frac{E}{\rho(1-\nu^2)}, \quad g^2 = \frac{D}{\rho h}.$$

Parameters c, k, $\chi$, $z_0$, $\rho$, E, $\nu$ and $\alpha$ are assumed to be constants.

Let us investigate the instance, when a change of density of sources along the radii obeys the Gaussian law, but their intensity changes in time according to the exponential law:

$$Q(r, t) = Q_0 \exp\left[-\left(\frac{r^2}{4r_0^2} + st\right)\right] \qquad t > 0. \tag{7.15}$$

Here $Q_0$, $r_0$ and s are constants. Substitution in formulas (7.12) yields:

$$T_0^*(p, t) = \frac{2Q_0 r_0^2 \exp(-p^2 r_0^2)}{c\rho k} \, \frac{\exp(-st) - \exp[-(\chi p^2 + a_0)t]}{\chi p^2 + a_0 - s},$$

$$Q_0^*(p, t) = \frac{24 Q_0 z_0 r_0^2 \exp(-p^2 r_0^2)}{c\rho k^3} \, \frac{\exp(-st) - \exp[-(\chi p^2 + a)t]}{\chi p^2 + a - s}.$$

For functions $u^*(p, t)$ and $w^*(p, t)$, applying formula (7.14) we obtain:

$$u^*(p,t) = \frac{2Q_0\alpha(1+\nu) g_0^2 r_0^2 p \exp(-p^2 r_0^2)}{c\rho k(\chi p^2 + a_0 - s)} \left\{ \frac{\exp(-st) - \cos g_0 pt + \frac{s}{g_0 p}\sin g_0 pt}{g_0^2 p^2 + s^2} - \frac{\exp[-(\chi p^2 + a_0)t] - \cos g_0 pt + \frac{\chi p^2 + a_0}{g_0 p}\sin g_0 pt}{g_0^2 p^2 + (\chi p^2 + a_0)^2} \right\}, \tag{7.16}$$

$$w^*(p,t) = \frac{24 Q_0 z_0 \alpha(1+\nu) g^2 r_0^2 p^2 \exp(-p^2 r_0^2)}{c\rho k^3 (\chi p^2 + a - s)} \left\{ \frac{\exp(-st) - \cos gp^2 t + \frac{s}{gp^2}\sin gp^2 t}{g^2 p^4 + s^2} - \frac{\exp[-(\chi p^2 + a)t] - \cos gp^2 t + \frac{\chi p^2 + a}{gp^2}\sin gp^2 t}{g^2 p^4 + (\chi p^2 + a)^2} \right\}. \tag{7.17}$$

Conversion of Hankle's transformation in this case cannot be carried out in the final form, therefore, integrals

$$u(p, t) = \int_0^\infty u^*(p, t) I_1(pr) p\,dp, \quad w(p, t) = \int_0^\infty w^*(p, t) I_0(pr) p\,dp$$

must be determined numerically.

Determination of conditions, under which it is possible to disregard the influence of inertial forces, considering the process to be quasi-stationary is of interest. Quasi-stationary solutions when t > 0 are yielded by formulas:

$$u_0^*(p, t) = \frac{2Q_0\alpha(1+\nu)}{c\rho h} \frac{r_0^2 \exp(-p^2 r_0^2)}{p} \frac{\exp(-st) - \exp[-(\chi p^2 + a_0)t]}{\chi p^2 + a_0 - s}, \tag{7.18}$$

$$w_0^*(p, t) = \frac{2Q_0 z_0 \alpha(1+\nu)}{c\rho h^3} \frac{r_0^2 \exp(-p^2 r_0^2)}{p^2} \frac{\exp(-st) - \exp[-(\chi p^2 + a)t]}{\chi p^2 + a - s}. \tag{7.19}$$

As it is noted in work [89], calculations show that longitudinal displacements can be found with sufficient accuracy by the formula (7.18), if we carry out the following conditions:

$$s r_0 \ll g_0, \quad \frac{\chi}{r_0^2} + a \ll g_0, \tag{7.20}$$

i.e., thermal processes occur sufficiently slowly as compared to the propagation of velocity of elastic waves.

Quasi-stationary approximation is unsuitable for finding of normal displacements. Actually, the integral

$$w_0(r, t) = \int_0^\infty w_0^*(p, t) I_0(pr) p\,dp$$

with function $w_0^*(p, t)$, determined according to (7.19), does not exist. At the same time the solution which takes into account inertial terms enables us to calculate the ultimate maximum of the temperature sag. Here it is essential that the plate is assumed to be of an unlimited magnitude. If the plate has finite dimensions, then quasi-stationary approximation becomes suitable for relatively slow thermal processes.

## § 8. Shells Exposed to Irradiation

First of all let us state several considerations, pertaining to the formulation of the problem. It is known that exposure of solids to irradiation is accompanied by numerous effects, as the result of which volume deformation appears in the solid body [91-93], and elastic and especially plastic characteristics of the substance are changed [94].

A neutron, possessing sufficient kinetic energy, passing through a crystal lattice, will form on its way primary, secondary, etc. recoil atoms. Atoms, knocked out from a crystal lattice leave vacant places and finally come to rest in internodes, which leads to formation in the lattice of paired Frenkel's "interstitial vacancy." An atom may be knocked out from a node, when it receives a certain threshold energy $E_d$. If the atom receives an energy, smaller than $E_d$, then this energy is dispersed for the excitation of lattice vibration (heating) without formation of displacements in it [95-98].

Interaction of neutrons with nuclei, in addition to elastic scattering can be accompanied by capture of neutrons and nuclear fission. With every act of disintegration energy is produced and new chamical elements are formed [99-102].

We will consider an initially uniform isotropic body, occupying a half-space $z \geq 0$. If neutrons fall on boundary $z = 0$ parallel to z axis with identical average energy and intensity $I_0 \frac{\text{neutron}}{\text{cm}^2 \cdot \text{sec}}$, then through simple reasoning we can find the intensity of the neutron flux, reaching plane $z = \text{const}$: the fall of flux $dI$ in layer $dz$ is proportional to $I(z)$ and $dz$; hence

$$I(z) = I_0 e^{-\mu z} \frac{\text{neutron}}{\text{cm}^2 \cdot \text{sec}}. \tag{8.1}$$

Value $\mu$ is called the macroscopic effective section. For any chemical element [98]

$$\mu = \sigma n_0 = \sigma \frac{A_0 \rho}{A} \tag{8.2}$$

and is of the order of $\frac{1}{cm}$, where $\sigma$ is the effective section, referred to one nucleus; $\rho$ is the density; A is the atomic weight; $A_0$ is Avogadro's number, and $n_0$ is the number of nuclei in 1 $cm^3$.

If $I_0$ does not depend on time, then by the moment of time through section z, will pass the flux

$$tI(z) = I_0 t e^{-\mu z}. \tag{8.3}$$

In the rough approximation we may assume that the change of volume of a substance, i.e., the cubic expansion $\theta$, is directly proportional to the flux $tI(z)$ and, consequently,

$$\theta = BI_0 t e^{-\mu z}, \tag{8.4}$$

where B is the experimental constant.

Value $I_0 t$ gives the total neutron flux per 1 $cm^2$ of the surface body. In reactors $I_0$ is of the order of $10^{13} - 10^{14}$ $\frac{neutrons}{cm^2 \cdot sec}$, and $nvt = I_0 t$ attains values of $10^{19} - 10^{23}$ $\frac{neutrons}{cm^2}$ while $\theta$ attains values of the order 0.1. Consequently, depending upon the energy of neutrons and irradiated material, value B may be of the order of $10^{23} - 10^{21}$ $\frac{cm^2}{neutron}$. Thus, for an estimate of the volume change we have:

$$\theta = Bnvte^{-\mu z}. \tag{8.5}$$

The relationship between the elastic modulus, yield point, ultimate strength, and the entire diagram of extension from nvt of various energies was investigated experimentally after irradiation of samples in nuclear reactors.

As numerous experiments show, upon the exposure of materials to irradiation, as a rule, the elastic modulus is changed very little (increases by 1.5-5% relative to the nonirradiated); with respect to the ultimate strength and the yield point, they are very sensitive to irradiation and especially the yield point [94].

For solid bodies with a plane boundary the number of neutrons passing at depth z under this boundary during the time t, will be expressed through flux nvt on the plane boundary according to formula

$$(nvt)_z = nvte^{-\mu z}, \tag{8.6}$$

and therefore, yield point $\sigma_s$ and shear modulus G will be variables according to depth z.

Let us introduce the hypothesis that properties, appearing at depth z coincide with properties during exposure to uniform irradiation with a power (nvt) [86]. Then diagrams for G and $\sigma_s$ with respect to nvt, analytically recorded in the form

$$G = G(nvt), \quad \sigma_s = \sigma_s(nvt),$$

yield curves of change of G and $\sigma_s$ throughout the depth z with a given nvt = N on the surface

$$G = G(Ne^{-\mu z}), \quad \sigma_s = \sigma_s(Ne^{-\mu z}).$$

If we subject to irradiation by power nvt = N, bodies with plane boundaries, a hollow cylinder or a sphere, the volume change being insignificant, then the distribution of stresses and deformations in them can be found by formulas of elastoplastic deformations of a heterogeneous body (see § 6 of this chapter).

Formula (8.6) assumes that neutron scattering $dN_z$ in layer dz is proportional to $N_z$ and to the thickness of layer dz, $dN_z = -\mu N_z dz$. In a radial flux from the inner surface, for instance, for a cylinder, this relationship is replaced by $d(rN_r) = -\mu r N_r dz$, and for a sphere,

with a radial flux from within, by $d(r^2N_r) = -\mu r^2 N_r dz$.

Thus, in the case of a hollow cylinder

$$N_r = N\frac{a}{r}e^{-\mu(r-a)}, \tag{8.7}$$

in the case of a sphere

$$N_r = N\frac{a^2}{r^2}e^{-\mu(r-a)}, \tag{8.8}$$

where N is the flux nvt per one unit of area of the inner surface of a cylinder or sphere (r = a). Consequently, for a cylinder

$$G = G\left[N\frac{a}{r}e^{-\mu(r-a)}\right], \quad \sigma_s = \sigma_s\left[N\frac{}{r}e^{-\mu(r-a)}\right], \tag{8.9}$$

for a sphere

$$G = G\left[N\frac{a^2}{r^2}e^{-\mu(r-a)}\right], \quad \sigma_s = \sigma_s\left[N\frac{a^2}{r^2}e^{-\mu(r-a)}\right]. \tag{8.10}$$

If, taking the above into account, we consider that elastic properties of metals are little changed during irradiation and, on the contrary, the yield point is changed essentially, then to calculate stresses and deformations of various kinds of shells, subjected to irradiation from the external or internal surface, we can apply the usual theory of elastic shells. Its specific character will consist of determination of loads, under which, for the first time the plastic deformation will appear, i.e., in the strength criteria.

Now we will set up the problem on exposure to irradiation of a shell and will indicate the course of solution. Let us assume that (x, y, z) is the accompanying Darboux trihedron on the middle surface of the shell, while x and y axes are directed along the main lines of curvature $\alpha$, $\beta$ and z – normal; $\varepsilon_1$, $\varepsilon_2$, $\gamma = 2\varepsilon_{12}$ are deformations of the middle surface; $\varkappa_1$, $\varkappa_2$, $\tau = \varkappa_{12}$ are changes of curvature and torsion through the action of external loads on the shell. Then the resultant forces and moments will be connected with values $\varepsilon$ and $\varkappa$ by

known linear relationships, while the intensity of deformations $e_i$ at a distance z from the middle surface in any one point of it will be

$$e_i = \frac{2}{\sqrt{3}} \sqrt{P_\varepsilon - 2zP_{\varepsilon\varkappa} + z^2 P_\varkappa}, \tag{8.11}$$

the intensity of stresses

$$\sigma_i = 3Ge_i, \tag{8.12}$$

while $P_\varepsilon$, $P_\varkappa$, $P_{\varepsilon\varkappa}$ are known quadratic forms for parameters $\varepsilon$ and $\varkappa$ [2].

Let us assume that the neutron flux proceeded normally to one of the surfaces of the shell (for instance, to the inner surface), while the total flux N = nvt is known as a function of coordinates of the middle surface (for instance, it is constant over the entire surface). To be specific let us assume that flux N is directed from the part of the inner surface $z = +\frac{h}{2}$, where h is the thickness of the shell. Then in layer z = const flux $N_z$ according to (8.6) will be

$$N_z = Ne^{-\mu\left(\frac{h}{2} - z\right)}. \tag{8.13}$$

Yield point $\sigma_s$, which is a known function of $N_z$,

$$\sigma_s = \sigma_s(N_z),$$

is, consequently, a known function of z and curvilinear coordinates of the shell.

Inasmuch as $\sigma_i = 3Ge_i$ as a result of elastic calculation of the shell is also a known function of coordinates, we can set up the difference

$$T = \sigma_s(N_z) - 3Ge_i =$$

$$= \sigma_s\left[Ne^{-\mu\left(\frac{h}{2} - z\right)}\right] - 2\sqrt{3}G\sqrt{P_\varepsilon - 2zP_{\varepsilon\varkappa} + z^2 P_\varkappa}. \tag{8.14}$$

We will designate with letter p the parameter, characterizing the load on the shell (for instance, the pressure on the surface), while owing

to linearity of the problem values $\varepsilon$ and $\varkappa$ will be proportional to p, and form P is proportional to the square of p. Let us designate

$$P_{\varepsilon}=p^2\tilde{P}_{\varepsilon},\quad P_{\varkappa}=p^2\bar{P}_{\varkappa},\quad P_{\varepsilon\varkappa}=p^2\tilde{P}_{\varepsilon\varkappa},\tag{8.15}$$

so that $\tilde{P}$ does not depend on p and they are known functions of curvilinear coordinates. Then function T is written in the form:

$$T=\sigma_s[N_z e^{-\mu\left(\frac{h}{2}-z\right)}]-2\sqrt{3}Gp\sqrt{\tilde{P}_{\varepsilon}-2z\tilde{P}_{\varepsilon\varkappa}+z^2P_{\varkappa}}\tag{8.16}$$

and is a linear function of p and a known function of coordinates $\alpha$, $\beta$, and z:

$$T=T(\alpha,\beta,z,p).$$

Let us set up a problem: to find that value of load $p_*$, with which in the shell, in some point $M^*(z = z_*,\ \alpha = \alpha_*,\ \beta = \beta_*)$ fluidity appears for the first time, i.e., $\sigma_1 = \sigma_s$.

Inasmuch as usually $\sigma_1$ in elastic shells attains a maximum when $z = \pm\frac{h}{2}$, one would think, that fluidity also appears for the first time somewhere on the external or internal surface. But, on the other hand, radiation strengthening will be the greatest precisely on one of these surfaces, and therefore, the appearance of fluidity on the irradiation surface becomes less probable.

In general, if point $M^*$ is inside the body of the shell (i.e., $|z| < \frac{h}{2}$ and $\alpha_*$, $\beta_*$ not on the boundary of the shell), conditions, determining $p_*$ and point $M^*$, have to have the form:

$$T=\frac{\partial T}{\partial\alpha}=\frac{\partial T}{\partial\beta}=\frac{\partial T}{\partial z}=0.\tag{8.17}$$

The problem is greatly simplified, if flux N is constant on the surface, i.e., $\frac{\partial N}{\partial\alpha}=\frac{\partial N}{\partial\beta}=0$. Actually, in this case condition $\frac{\partial T}{\partial\alpha}=\frac{\partial T}{\partial\beta}=0$ coincides with condition $\frac{\partial e_i}{\partial\alpha}=\frac{\partial e_i}{\partial\beta}=0$, i.e., with the

usual in the theory of shells condition of detecting the point of maximum stresses, and, consequently $\alpha_*$, $\beta_*$, become known on the basis of the usual method for finding them. After this, it remains to find only the coordinate of layer $z = z_*$, where fluidity begins. For that, from (8.13) we find z through $N_z$:

$$z = \frac{h}{2} + \frac{1}{\mu} \ln \frac{N_z}{N}. \tag{8.18}$$

Now, considering that in (8.16) z is replaced by this expression, conditions $T = \frac{\partial T}{\partial z} = 0$ are written in the form $\left(\frac{dz}{dN_z} = \frac{1}{\mu N_z}\right)$, first equations for $z_*$:

$$z = z_*, \quad \varphi(N_z) = \psi(N_z), \tag{8.19}$$

where

$$\varphi(N_z) = \frac{N_z}{\sigma_s(N_z)} \frac{d\sigma_s(N_z)}{dN_z},$$

$$\psi(N_z) = \frac{1}{\mu} \frac{-\tilde{p}_{12} + z\tilde{p}_2}{\tilde{p}_1 - 2z\tilde{p}_{12} + z^2\tilde{p}_2}, \tag{8.20}$$

while z has value (8.18), and, secondly, expression for $p_*$:

$$\tilde{p}_* = \frac{\sigma_s(N_{z*})}{2\sqrt{3G}\sqrt{\tilde{p}_1 - 2z_*\tilde{p}_{12} + z_*^2\tilde{p}_2}}. \tag{8.21}$$

If point $M^*$ lies on surface $\left(z = \pm\frac{h}{2}\right)$, the problem is solved by one equation $T = 0$.

Equation (8.19) is solved graphically. Point of intersection of curves $\varphi(N)$ and $\psi(N)$ yields $N_*$ and $\varphi(N_*)$. After that we find $\sigma_s(N_*)$ and $z_*$ from equation (8.18), and then from (8.21) we determine $p_*$, with which the plastic deformation of shell begins.

It can happen that curves $\varphi$ and $\psi$ have several points of intersection $N_*'$, $N_*''$ ..., and to each of them will correspond its own $p_*'$, $p_*''$, ....

Now the problem on where the plastic deformations appear for the first time will be solved by means of determination of the least of the values of $p_*$ found above.

Formulas of calculation of the strength of bodies under the effect of exposure to irradiation will include the dimensional physical constant $[\mu] \sim \frac{1}{cm}$, and therefore, geometrically and mechanically similar bodies from identical material are absolutely unequally strong.

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# ERRATA

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$$T_1 = \frac{3D}{h^2}(\varepsilon_1 + \nu\varepsilon_2), \quad T_2 = \frac{3D}{h^2}(\varepsilon_2 + \nu\varepsilon_1).$$

In order to satisfy equation $\frac{\partial T_1}{\partial x} = 0$ and condition $T_1 = 0$ when $x = \pm l$, we must assume that $T_1 = 0$ or $\varepsilon_1 = -\nu\varepsilon_2$, from this it follows that

$$T_2 = 3D(1-\nu^2)\frac{\varepsilon_2}{h^2}.$$

The third equation of equilibrium is reduced to the form

$$-\frac{\partial^4 w}{\partial x^4} + \frac{3(1-\nu^2)}{a^2h^2}(c-w) = 0,$$

and boundary conditions assume the form

$$-\frac{\partial^2 w}{\partial x^2} + \frac{\nu c}{a^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0 \quad \text{when } x = \pm l.$$

Presenting c - w in the form of the sum of terms of type $\zeta e^{mx}$, we shall find that $m^2$ will be a value of the order of $\frac{1}{h}$, and the solution will assume the form:

$$w = c + c_1 \operatorname{ch}\left(\frac{qx}{a}\right)\cos\left(\frac{qx}{a}\right) + c_2 \operatorname{sh}\left(\frac{qx}{a}\right)\sin\left(\frac{qx}{a}\right), \tag{2.17}$$

where

$$q^2 = \frac{a}{2h}\sqrt{3(1-\nu^2)}$$

and

$$c_1 = -\frac{\nu c}{q^2}\,\frac{\operatorname{sh}\left(\frac{ql}{a}\right)\cos\left(\frac{ql}{a}\right) - \operatorname{ch}\left(\frac{ql}{a}\right)\sin\left(\frac{ql}{a}\right)}{\operatorname{sh}\left(\frac{2ql}{a}\right) + \sin\left(\frac{2ql}{a}\right)},$$

$$c_2 = -\frac{\nu c}{q^2}\,\frac{\operatorname{sh}\left(\frac{ql}{a}\right)\cos\left(\frac{ql}{a}\right) + \operatorname{ch}\left(\frac{ql}{a}\right)\sin\left(\frac{ql}{a}\right)}{\operatorname{sh}\left(\frac{2ql}{a}\right) + \sin\left(\frac{2ql}{a}\right)}. \tag{2.18}$$

The solution shows that near the edges of values $\varepsilon_1$, $\varepsilon_2$, $h\varkappa_1$, and $h\varkappa_2$ are all of the same order; at distances from ends, which exceed the value (ah)1/2, $\varepsilon_1$ and $\varepsilon_2$ become small as compared to $h\varkappa_2$.

We can show that in a given static problem the potential extension energy will be of the order of the value equal to the product

of $(\frac{h}{a})^{1/2}$ multiplied by the potential bending energy. In case of oscillations we may conclude that extension, which ensures the fulfillment of conditions on the edges, is limited in actual practice by a narrow ban along the edges. The changes in the total potential energy and oscillation period connected with this are so small that they can be disregarded.

## §3. The Spherical Shell

The mathematical aspect of the investigation of oscillations of the spherical shell is similar to the investigation of the cylindrical shell, and therefore, there is no need to reproduce the analogous formulas here. Let us examine the qualitative aspect of elastic oscillations of the spherical shell.

Let us assume that oscillations are accompanied by elongations; they are divided into two classes, which are obtained by rejecting the radial component of displacement and the radial component of rotation, respectively. With every oscillation of the normal type, belonging to either class, displacements are expressed by means of spherical functions of any specific integral order. With oscillations of the first class the frequency $\frac{p}{2\pi}$ is connected with the order of the spherical function n by the relationship

$$\frac{p^2 a^2 \rho}{G} = (n-1)(n+2), \tag{3.1}$$

where a is the radius of the sphere. In oscillations of the second class the analogous relationship will be

$$\frac{p^4 a^4 \rho^2}{G^2} - \frac{p^2 a^2 \rho}{G}\left[(n^2+n+4)\frac{1+\nu}{1-\nu} + (n^2+n-2)\right] + 4(n^2+n-2)\frac{1+\nu}{1-\nu} = 0. \tag{3.2}$$